DISPLAY MAPS IN CAUCHY COMPLETE CATEGORIES

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Abstract. It has been known that categorical interpretations of dependent type theory with Σ and $\mathtt{Id}$ types induce weak factorization systems. When one has a weak factorization system $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{C}$ in hand, it is then natural to ask whether or not $(\mathcal{L}, \mathcal{R})$ harbors an interpretation of dependent type theory with Σ and $\mathtt{Id}$ (and possibly Π) types. Using the framework of *categories of display maps* to phrase this question more precisely, one would ask whether or not there exists a class $\mathcal{D}$ of morphisms of $\mathcal{C}$ such that (1) the retract closure of $\mathcal{D}$ is the class $\mathcal{R}$ and (2) the pair $(\mathcal{C}, \mathcal{D})$ forms a category of display maps modeling Σ and $\mathtt{Id}$ (and possibly Π) types. In this paper, we show, with the hypothesis that $\mathcal{C}$ is Cauchy complete, that there exists such a class $\mathcal{D}$ if and only if $(\mathcal{C}, \mathcal{R})$ *itself* forms a category of display maps modeling Σ and $\mathtt{Id}$ (and possibly Π) types. Thus, we reduce the search space of our original question from a potentially proper class to a singleton.

Contents

1. Introduction

It has long been known that categorical interpretations (in, for example, categories of display maps) of dependent type theory with Σ and $\mathtt{Id}$ types induce weak factorization systems (see, for example, [GG08]). Thus, a search for such interpretations could comprise two steps: first, identify a weak factorization system $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{C}$, and second, decide if $(\mathcal{L}, \mathcal{R})$ harbors an interpretation of dependent type theory with Σ and $\mathtt{Id}$ types. We are interested in the connection between dependent type theory and familiar weak factorization systems of homotopy theory, so we are interested primarily in the second step.

This is the first in a series of papers in which we develop a theorem for recognizing whether a given weak factorization system $(\mathcal{L}, \mathcal{R})$ harbors a model of dependent type theory with Σ and $\mathtt{Id}$ (and possibly Π) types. (The content of this series can already be found in Chapters 1-3 of [Nor17].)

In this and following papers, we choose *display map categories* from the various categorical frameworks which can interpret dependent type theory. This is because the data of a display map category, which consists of a category $\mathcal{C}$ and a class of maps of $\mathcal{C}$, is directly comparable to the data underlying a weak factorization system, which consists of a category $\mathcal{C}$ and two classes of maps of $\mathcal{C}$ (each of which determine the other).

Not only do we choose the simplest categorical framework for interpreting dependent type theory, but we have also chosen the simplest variant of *weak factorization system* (compared to, for example, *algebraic* weak factorization systems). We make these choices in order to reveal the most fundamental connection between these two notions. In future work, we hope to build on this by studying the connections between more structure notions in both instances, say that of comprehension categories and algebraic weak factorization systems.

In this paper, we study categories of display maps $(\mathcal{C}, \mathcal{D})$ which model Σ and $\mathtt{Id}$ (and possibly Π) types. As mentioned above (and shown below in Proposition 3.2), such a structure generates a weak factorization system $({}^{\boxslash}\mathcal{D}, \overline{\mathcal{D}})$ on the category $\mathcal{C}$ where $\overline{\mathcal{D}}$ is the retract closure of $\mathcal{D}$. In this framework, for any weak factorization system $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{C}$, our original question,

Question 1.1. *Does $(\mathcal{L}, \mathcal{R})$ harbor an interpretation of dependent type theory with Σ and $\mathtt{Id}$ (and possibly Π) types?*

can be phrased more precisely as the following question.

Question 1.2. *Does there exist a subclass $\mathcal{D} \subseteq \mathcal{R}$ such that $\mathcal{R}$ is the retract closure of $\mathcal{D}$ and $(\mathcal{C}, \mathcal{D})$ is a display map category which models Σ and $\mathtt{Id}$ (and possibly Π) types?*

The original contribution of this paper is Theorem 4.13 which says that when $\mathcal{C}$ is Cauchy complete, Question 1.2 is equivalent to the following question.

Question 1.3. *Is $(\mathcal{C}, \mathcal{R})$ is a display map category with Σ and $\mathtt{Id}$ (and possibly Π) types?*

Thus, to decide whether or not $(\mathcal{L}, \mathcal{R})$ harbors an interpretation, we do not have to analyze the pair $(\mathcal{C}, \mathcal{D})$ for all classes $\mathcal{D}$ whose retract closure is $\mathcal{R}$, which likely constitute a proper class. Rather, we only need to analyze the one pair $(\mathcal{C}, \mathcal{R})$. Indeed, this will be the task of the following papers of this series.

2. Display map categories.

In this section, we fix definitions of *display map categories* and of Σ, $\mathtt{Id}$, and Π types in display map categories.

Definition 2.1. A *display map category* $(\mathcal{C}, \mathcal{D})$ consists of a category $\mathcal{C}$ with a terminal object and a class $\mathcal{D}$ of morphisms of $\mathcal{C}$ such that:

(1) $\mathcal{D}$ contains every isomorphism;
(2) $\mathcal{D}$ contains every morphism whose codomain is the terminal object;
(3) every pullback of every morphism of $\mathcal{D}$ exists; and

(4) $\mathcal{D}$ is stable under pullback.

We call the elements of $\mathcal{D}$ *display maps*.

If our display maps were required to be closed under composition, then this would coincide with Joyal's notion of *tribe* [Joy13]. Without conditions (1) and (2), this is the definition of a *class $\mathcal{D}$ of displays* in $\mathcal{C}$ [Tay99, Def. 8.3.2].

It is well known that a display map category gives rise to a comprehension category. We make a careful comparison of our display map categories and the comprehension categories of [Jac93] in Chapter 2 of [Nor17]: we show there that the Σ types defined below coincide with the strong sums of [Jac93], and that the Π types defined below coincide with the products of [Jac93].

In such a category of display maps, the objects of $\mathcal{C}$ are meant to represent contexts and the morphisms of $\mathcal{C}$ represent context morphisms. A morphism $p : E \to B$ of $\mathcal{D}$ represents a type family E dependent on B. The empty context is represented by the terminal object of $\mathcal{C}$, so condition (2) says that every object of $\mathcal{C}$ may be viewed as a type dependent on the empty context. The pullback of a morphism p of $\mathcal{D}$ along a morphism f of $\mathcal{C}$ represents the substitution of f into the type family p.

Definition 2.2. A category of display maps $(\mathcal{C}, \mathcal{D})$ *models Σ types* if $\mathcal{D}$ is closed under composition.

We call a composition gf of display maps a *Σ type* and sometimes denote it by $\Sigma_g f$.

Definition 2.3. A display map category $(\mathcal{C}, \mathcal{D})$ *models Π types* if for every pair of composable display maps $g : W \to X$ and $f : X \to Y$, there exists a display map $\Pi_f g$ with codomain Y and the universal property

$$\mathcal{C}/Y(y, \Pi_f g) \cong \mathcal{C}/X(f^* y, g)$$

natural in y.

The term *Π type* will refer to such a display map $\Pi_f g$.

Now we define $\mathtt{Id}$ types in a display map category. This definition is more convoluted and less standard than the preceding definitions, but in Remark 2.6 below, we justify this choice of definition by comparing it with others.

First we fix some notation.

Notation 2.4. For a class $\mathcal{M}$ of morphisms of a category $\mathcal{C}$, let ${}^{\boxslash}\mathcal{M}$ denote the class of morphisms of $\mathcal{C}$ which have the left lifting property against $\mathcal{M}$. Similarly, let $\mathcal{M}^{\boxslash}$ denote the class of morphisms of $\mathcal{C}$ which have the right lifting property against $\mathcal{M}$.

Definition 2.5. Consider a category of display maps $(\mathcal{C}, \mathcal{D})$ which models Σ types. We say that it *models $\mathtt{Id}$ types* if for every $f : X \to Y$ in $\mathcal{D}$, the diagonal $\Delta_f : f \to f \times f$ in $\mathcal{C}/Y$ has a factorization $\Delta_f = \epsilon_f r_f$

$$\begin{tikzcd}
X \arrow[r, "r_f"] \arrow[dr, "f"'] & \mathtt{Id}(f) \arrow[r, "\epsilon_f"] \arrow[d, "\iota_f"] & X \times_Y X \arrow[dl, "f \times f"] \\
& Y &
\end{tikzcd}$$

in $\mathcal{C}/Y$ where ϵ_f is in $\mathcal{D}$ and for every morphism $\alpha : A \to X$ in $\mathcal{C}$, the pullback $\alpha^* r_f$, as shown below, is in ${}^{\boxslash}\mathcal{D}$ for each $i = 0, 1$.

(∗)
$$\begin{tikzcd}
& \alpha^*\mathtt{Id}(f) \arrow[rr] \arrow[dd] & & \mathtt{Id}(f) \arrow[dd, "\pi_i\epsilon_f"] \\
A \arrow[ur, "\alpha^* r_f"] \arrow[rr] \arrow[dr, equal] & & X \arrow[ur, "r_f"] \arrow[dr, equal] & \\
& A \arrow[rr, "\alpha"] & & X
\end{tikzcd}$$

We will call the morphism $\iota_f : \mathtt{Id}(f) \to Y$ the $\mathtt{Id}$ *type of* f in $\mathcal{C}/Y$.

Note that since $(\mathcal{C}, \mathcal{D})$ models Σ types in this definition, $\mathcal{D}$ is closed under composition and is stable under pullback. Thus, for any $f \in \mathcal{D}$, $f \times f$ is in $\mathcal{D}$ as it is the composition of a pullback of f with f, and ι_f is in $\mathcal{D}$ since it is the composition of ϵ_f and $f \times f$.

Remark 2.6. This definition is slightly stronger than that which is usually given for $\mathtt{Id}$ types. Usually, only pullbacks of r of the following form

(∗∗)
$$\begin{tikzcd}
& \alpha^*\mathtt{Id}(f) \arrow[rr] \arrow[dd] & & \mathtt{Id}(f) \arrow[dd, "\iota_f"] \\
\alpha^* f \arrow[ur, "\alpha^* r_f"] \arrow[rr] \arrow[dr] & & X \arrow[ur, "r_f"] \arrow[dr, "f"] & \\
& A \arrow[rr, "\alpha"] & & Y
\end{tikzcd}$$

are required to be in ${}^{\boxslash}\mathcal{D}$. Since the map r_f is defined 'in the context Y', this can be interpreted as ensuring that the property $r_f \in {}^{\boxslash}\mathcal{D}$ is stable under any substitution $\alpha : A \to Y$. For example, this is how the *weakly stable identity types* of [LW15] are defined.

For the moment, we will denote by P_* (respectively, P_{**}) the property that the pullbacks of r_f of the form shown in diagram (∗) (respectively, (∗∗)) are in ${}^{\boxslash}\mathcal{D}$.

To see that P_* is stronger than P_{**}, consider the situation displayed in diagram (∗∗) and assume that P_* holds. We can obtain the morphism $\alpha^* r_f$ in diagram (∗∗) by first pulling α back along f and then pulling r_f back along $f^*\alpha$, as shown below in diagram (†).

(†)
$$\begin{tikzcd}
& \alpha^*\mathtt{Id}(f) \arrow[rr] \arrow[dd] & & \mathtt{Id}(f) \arrow[dd, "\pi_i\epsilon_f"] \\
A \arrow[ur, "\alpha^* r_f"] \arrow[rr] \arrow[dr, equal] & & X \arrow[ur, "r_f"] \arrow[dr, equal] & \\
& \alpha^* A \arrow[rr, "f^*\alpha"] \arrow[d] \arrow[drr, phantom, very near start, "\lrcorner"] & & X \arrow[d, "f"] \\
& A \arrow[rr, "\alpha"] & & Y
\end{tikzcd}$$

Since the triangular prism on the top of diagram (†) is of the form of that in diagram (∗), $\alpha^* r_f$ is in ${}^{\boxslash}\mathcal{D}$. Thus, P_{**} holds.

The stronger property P_* entails a common variant of path induction. When the α of diagram $(*)$ is a point $* \to X$, then this is called *based path induction* or *Paulin-Mohring elimination*, and, in fact, it is equivalent to path induction in the presence of Π types (see [Uni13, §1.12.2] or [Str93], where the crux of the proof first appeared as a theorem due to Martin Hofmann). Then the property P_* itself can be understood as a 'parametrized' version of based path induction.

In [GG08] and [BG12], the authors also work with a stronger variant of `Id` types, called strong `Id` types in [BG12]. Lemma 11 of [GG08] shows that our definition of `Id` types follows from theirs, and [Nor17, Proposition A.2.2] shows that their definition follows from ours.

In summary, our definition of `Id` types is not inconsistent with what is already in the literature. Our reason for using this stronger definition, though, is that it is precisely what is necessary to generate a weak factorization system.

To be more precise, let $\alpha_i^* r_f$ denote the pullback in diagram $(*)$ for $i = 0, 1$. We will see in Proposition 3.2 that requiring that all $\alpha_0^* r_f$ are in ${}^{\boxslash}\mathcal{D}$ is exactly the requirement that the left factor of the factorization given there is in ${}^{\boxslash}\mathcal{D}$, the left class of our weak factorization system. Then the requirement that each $\alpha_1^* r_f$ is in ${}^{\boxslash}\mathcal{D}$ might be justified by a desire for symmetry, but it is actually a consequence of requiring that all $\alpha_0^* r_f$ are in ${}^{\boxslash}\mathcal{D}$ ([Nor17, Lemma 3.3.9]).

The definition we have given here of `Id` types treats each slice $\mathcal{C}/Y$ of $\mathcal{C}$ equally, as the syntax of type theory does. However, in the next section, we will find it useful to consider only the `Id` types in the slice $\mathcal{C}/* \cong \mathcal{C}$.

Definition 2.7. Consider a category $(\mathcal{C}, \mathcal{D})$ of display maps which models Σ types. We say that it *models* `Id` *types of objects* if it has all `Id` types in $\mathcal{C}/* \cong \mathcal{C}$. That is: if for every X in $\mathcal{C}$, the diagonal $\Delta_X : X \to X \times X$ has a factorization

$$X \xrightarrow{r_X} \mathtt{Id}(X) \xrightarrow{\epsilon_X} X \times X$$

in $\mathcal{C}$ where ϵ_X is in $\mathcal{D}$ and for every morphism $\alpha : A \to X$ in $\mathcal{C}$, the pullback $\alpha^* r_X$, as shown below, is in ${}^{\boxslash}\mathcal{D}$ for each $i = 0, 1$.

$$\begin{tikzcd}
& \alpha^*\mathtt{Id}(X) \arrow[rr] \arrow[dd] & & \mathtt{Id}(X) \arrow[dd, "\pi_i \epsilon_X"] \\
A \arrow[ur, "\alpha^* r_X"] \arrow[rr] \arrow[dr, equal] & & X \arrow[ur, "r_X"] \arrow[dr, equal] & \\
& A \arrow[rr, "\alpha"] & & X
\end{tikzcd}$$

3. Weak factorization systems from display map categories.

In this section, we show how any display map category $(\mathcal{C}, \mathcal{D})$ with Σ types and `Id` types on objects generates a weak factorization system $({}^{\boxslash}\mathcal{D}, \overline{\mathcal{D}})$ with a factorization (λ, ρ) where $\overline{\mathcal{D}}$ is the retract closure of $\mathcal{D}$ and the image of ρ lies in $\mathcal{D}$. This will give us a good enough handle on the relationship between $\mathcal{D}$ and $\overline{\mathcal{D}}$ to prove our main theorem in the following section, where we extend an interpretation of type theory in $(\mathcal{C}, \mathcal{D})$ to one in $(\mathcal{C}, \overline{\mathcal{D}})$.

In the following theorem, we construct this weak factorization system. The proof uses ideas from the proof of Theorem 10 of [GG08], where a weak factorization system is constructed in the classifying category of a dependent type theory. A categorical version appears as Theorem 2.8 in [Emm14].

Notation 3.1. Let $\overline{\mathcal{D}}$ denote $({}^{\boxslash}\mathcal{D})^{\boxslash}$.

Proposition 3.2. *Consider a category of display maps $(\mathcal{C}, \mathcal{D})$ which models Σ types and $\mathtt{Id}$ types of objects. There exists a weak factorization system $({}^{\boxslash}\mathcal{D}, \overline{\mathcal{D}})$ in $\mathcal{C}$ with a factorization (λ, ρ) where the image of ρ is contained in $\mathcal{D}$.*

Proof. The factorization is defined in the following way for any $f : X \to Y$ in $\mathcal{C}$. We have a factorization

$$Y \xrightarrow{r_Y} \mathtt{Id}(Y) \xrightarrow{\epsilon_Y} Y \times Y$$

of the diagonal $\Delta : Y \to Y \times Y$. Now we define the factorization of f to be

$$X \xrightarrow{1 \times r_Y f} X \times_Y \mathtt{Id}(Y) \xrightarrow{\pi_1 \epsilon_Y} Y$$

where the middle object is obtained in the following pullback.

$$\begin{tikzcd}
X \times_Y \mathtt{Id}(Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathtt{Id}(Y) \arrow[d, "\pi_0 \epsilon_Y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

The left factor

$$\lambda(f) := 1 \times r_Y f : X \to X \times_Y \mathtt{Id}(Y)$$

is obtained as the following pullback of r_Y.

$$\begin{tikzcd}
& f^*\mathtt{Id}(Y) \arrow[rr] \arrow[dd] & & \mathtt{Id}(Y) \arrow[dd, "\pi_0 \epsilon_Y"] \\
X \arrow[ur, "f^* r_Y"] \arrow[rr] \arrow[dr, equal] & & Y \arrow[ur, "r_Y"] \arrow[dr, equal] & \\
& X \arrow[rr, "f"] & & Y
\end{tikzcd}$$

Thus, it is in ${}^{\boxslash}\mathcal{D}$.

The right factor

$$\rho(f) := \pi_1 \epsilon_Y : X \times_Y \mathtt{Id}(Y) \to Y$$

is in $\mathcal{D}$ because it is the composition of a pullback of ϵ_Y with a pullback of $X \to *$.

$$\begin{tikzcd}
X \times_Y \mathtt{Id}(Y) \arrow[r] \arrow[dd] \arrow[ddd, bend right, "\rho(f)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathtt{Id}(Y) \arrow[d, "\epsilon_Y"] \\
& Y \times Y \\
X \times Y \arrow[ur, "f \times 1"] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\
Y \arrow[r] & *
\end{tikzcd}$$

Now any morphism in $\mathcal{D}$ has the right lifting property against every morphism with the left lifting property against $\mathcal{D}$. Thus $\mathcal{D} \subseteq ({}^{\boxslash}\mathcal{D})^{\boxslash}$, and we conclude that each $\rho(f) \in ({}^{\boxslash}\mathcal{D})^{\boxslash}$.

Now it remains to check that ${}^{\boxslash}\mathcal{D} = {}^{\boxslash}(({}^{\boxslash}\mathcal{D})^{\boxslash})$. Clearly, every morphism in ${}^{\boxslash}\mathcal{D}$ has the left lifting property against every morphism with the right lifting property against ${}^{\boxslash}\mathcal{D}$. Therefore, ${}^{\boxslash}\mathcal{D} \subseteq {}^{\boxslash}(({}^{\boxslash}\mathcal{D})^{\boxslash})$. We showed in the last paragraph that $\mathcal{D} \subseteq ({}^{\boxslash}\mathcal{D})^{\boxslash}$, and applying ${}^{\boxslash}(-)$ to this, we obtain ${}^{\boxslash}\mathcal{D} \supseteq {}^{\boxslash}(({}^{\boxslash}\mathcal{D})^{\boxslash})$. Therefore, ${}^{\boxslash}\mathcal{D} = {}^{\boxslash}(({}^{\boxslash}\mathcal{D})^{\boxslash})$. $\square$

We will show that $\overline{\mathcal{D}}$ is the retract closure of $\mathcal{D}$, justifying its notation. First, we record a well-known lemma.

Lemma 3.3. Consider a category of display maps $(\mathcal{C}, \mathcal{D})$ which models Σ types and `Id` types of objects.

The class $\overline{\mathcal{D}}$ contains all isomorphisms, is closed under composition and retracts, and is stable under pullback.

Proof. This is true of any class of morphisms of the form $\mathcal{M}^{\boxslash}$ as shown in [MP12, Proposition 14.1.8]. We will just show here that $\overline{\mathcal{D}}$ is closed under retracts since this fact plays a central role in this paper.

Consider a morphism $g : W \to Z$ which is a retract of morphism $f : X \to Y$ in $\overline{\mathcal{D}}$.

$$\begin{tikzcd}
W \arrow[r, "w"'] \arrow[rr, bend left, equal] \arrow[d, "g"] & X \arrow[r, "x"'] \arrow[d, "f"] & W \arrow[d, "g"] \\
Z \arrow[r, "z"] \arrow[rr, bend right, equal] & Y \arrow[r, "y"] & Z
\end{tikzcd}$$

In order to show that g has the right lifting property against ${}^{\boxslash}\mathcal{D}$, consider any lifting problem as on the left below with $\ell \in {}^{\boxslash}\mathcal{D}$.

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "\ell"'] & W \arrow[d, "g"] \\
C \arrow[r, "c"] \arrow[ur, dashed] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "\ell"'] & W \arrow[r, "w"] \arrow[d, "g"'] & X \arrow[r, "x"] \arrow[d, "f"'] & W \arrow[d, "g"] \\
C \arrow[r, "c"'] \arrow[urr, dashed, "s"] & Z \arrow[r, "z"'] & Y \arrow[r, "y"] & Z
\end{tikzcd}$$

Then we can form the solid portion of the diagram on the right above by tacking the retract diagram onto our lifting problem. Since f is in $\overline{\mathcal{D}}$, is has the right lifting property against ℓ, so there exists a morphism $s : C \to X$. Then the composition $xs : C \to W$ is a solution to our original lifting problem.

Therefore, g is in $({}^{\boxslash}\mathcal{D})^{\boxslash} = \overline{\mathcal{D}}$. $\square$

Proposition 3.4. *Consider a category of display maps $(\mathcal{C}, \mathcal{D})$ which models Σ types and `Id` types of objects.*

The class $\overline{\mathcal{D}}$ is the retract-closure (in $\mathcal{C}^2$, the category of morphisms of $\mathcal{C}$) of $\mathcal{D}$. Moreover, every morphism $f : X \to Y$ of $\overline{\mathcal{D}}$ is a retract in $\mathcal{C}/Y$ of the morphism $\rho(f)$ defined in Proposition 3.2.

Proof. Since $\overline{\mathcal{D}}$ is the class of morphisms with the right lifting property against all morphisms with the left lifting property against $\mathcal{D}$, we have that $\mathcal{D} \subseteq \overline{\mathcal{D}}$. We showed above in Lemma 3.3 that $\overline{\mathcal{D}}$ is closed under retracts. Now we just need to show that any f in $\overline{\mathcal{D}}$ is a retract of a morphism in $\mathcal{D}$. Indeed, we show that it is a retract of $\rho(f)$, which we know to be in $\mathcal{D}$ by Proposition 3.2.

Consider any morphism $f : X \to Y$ of $\overline{\mathcal{D}}$ and the following lifting problem.

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "\lambda(f)"'] & X \arrow[d, "f"] \\
X \times_Y \mathtt{Id}(Y) \arrow[r, "\rho(f)"'] \arrow[ur, dashed] & Y
\end{tikzcd}$$

It has a solution $s : X \times_Y \mathtt{Id}(Y) \to X$ since $\lambda(f) \in {}^{\boxslash}\mathcal{D}$ and $f \in ({}^{\boxslash}\mathcal{D})^{\boxslash}$. Then we can rearrange the lifting problem diagram into the following commutative diagram.

$$\begin{tikzcd}
X \arrow[r, "\lambda(f)"'] \arrow[rr, equal, bend left] \arrow[dr, "f"'] & X \times_Y \mathtt{Id}(Y) \arrow[r, "s"'] \arrow[d, "\rho(f)"] & X \arrow[dl, "f"] \\
& Y &
\end{tikzcd}$$

This diagram shows f to be a retract of $\rho(f)$ in the category $\mathcal{C}/Y$, and thus also in the category $\mathcal{C}^2$. Therefore, any morphism of $\overline{\mathcal{D}}$ is a retract of a morphism in $\mathcal{D}$. $\square$

4. Cauchy complete categories.

In this section, we prove our main theorem: if a category $\mathcal{C}$ is Cauchy complete and $(\mathcal{C}, \mathcal{D})$ is a category of display maps which models Σ types and $\mathtt{Id}$ types, then $(\mathcal{C}, \overline{\mathcal{D}})$ is also a category of display maps which models Σ types and $\mathtt{Id}$ types. Moreover, if $(\mathcal{C}, \mathcal{D})$ also models Π types, then $(\mathcal{C}, \overline{\mathcal{D}})$ models Π types as well.

Note that in these results, the hypothesis that $\mathcal{C}$ is Cauchy complete is only necessary to establish that $(\mathcal{C}, \overline{\mathcal{D}})$ is a category of display maps and that $(\mathcal{C}, \overline{\mathcal{D}})$ models Π types when the same are true of $(\mathcal{C}, \mathcal{D})$. The result that $(\mathcal{C}, \overline{\mathcal{D}})$ models Σ types is shown using without this hypothesis, using a straightforward homotopical argument. The result that $(\mathcal{C}, \overline{\mathcal{D}})$ models $\mathtt{Id}$ types can also be shown using without this hypothesis, in a homotopical argument, as is done in [Nor17, Prop. A.1.5]. However, we have not yet developed the machinery required by that proof in this series of papers. Instead, we give here a proof which does require the hypotheses that $\mathcal{C}$ is Cauchy complete and that the $\mathtt{Id}$ types are given in a functorial way.

For those results which do utilize Cauchy completeness, the proofs use the following idea. In both cases, we need to prove that a certain functor, built out of elements of $\overline{\mathcal{D}}$, is representable while we hypothesize that the same functor, if built only out of elements of $\mathcal{D}$, is representable. In a Cauchy complete category, retracts of representable functors are themselves representable. Thus, using the fact that every element of $\overline{\mathcal{D}}$ is a retract of an element of $\mathcal{D}$, we aim to show that those functors we want to be representable are retracts of those functors we know to be representable.

4.1. Preliminaries.

In this section, we recall the basic definitions and results that are necessary for our narrative.

Definition 4.1 ([Bor94, Def. 6.5.8]). A category $\mathcal{C}$ is *Cauchy complete* if every idempotent splits: that is, for every idempotent $e : C \to C$ in $\mathcal{C}$, there is a retract of C

$$R \xrightarrow{i} C \xrightarrow{r} R$$

such that $ir = e$.

Lemma 4.2. If a category $\mathcal{C}$ is Cauchy complete, then every slice $\mathcal{C}/X$ for $X \in \mathcal{C}$ is Cauchy complete.

Proof. Consider an idempotent $c \to c$ in a slice $\mathcal{C}/X$ which is represented by the following diagram in $\mathcal{C}$.

$$(*) \qquad \begin{tikzcd} C \arrow[rr, "e"] \arrow[dr, "c"'] & & C \arrow[dl, "c"] \\ & X & \end{tikzcd}$$

Then the morphism $e : C \to C$ is an idempotent in $\mathcal{C}$. It splits into

$$R \xrightarrow{i} C \xrightarrow{r} R$$

such that $ir = e$. Since $cir = ce = c$, the following diagram commutes.

$$\begin{tikzcd} R \arrow[r, "i"] \arrow[dr, "ci"'] & C \arrow[r, "r"] \arrow[d, "c"] & R \arrow[dl, "ci"] \\ & X & \end{tikzcd}$$

This is a retraction in $\mathcal{C}/X$ which splits our original idempotent $(*)$. □

We will make extensive use of the following two lemmas so we record them here.

Lemma 4.3. Consider a category $\mathcal{C}$, idempotents $e : C \to C$ and $f : D \to D$ in $\mathcal{C}$, and a morphism $c : C \to D$ making the following diagram commute.

$$\begin{tikzcd} C \arrow[r, "e"] \arrow[d, "c"] & C \arrow[d, "c"] \\ D \arrow[r, "f"] & D \end{tikzcd}$$

Then splittings of both $e : C \to C$ and $f : D \to D$ extend uniquely to a splitting of the idempotent $\langle e, f \rangle$ in $\mathcal{C}^2$.

$$\begin{tikzcd} R \arrow[r, "i"] \arrow[d, dashed, "sci"'] & C \arrow[r, "r"] \arrow[d, "c"] & R \arrow[d, dashed, "sci"] \\ S \arrow[r, "j"] & D \arrow[r, "s"] & S \end{tikzcd}$$

Moreover, if c is an isomorphism, then so is sci.

Proof. The proof is a straightforward diagram chase. □

Corollary 4.4. *If $\mathcal{C}$ is Cauchy complete, then $\mathcal{C}^2$ is Cauchy complete.*

Corollary 4.5. *Splittings of idempotents are unique up to unique isomorphism.*

This result can be used to show that a splitting of idempotent $e : C \to C$ can be obtained as either the equalizer or the coequalizer of the diagram $1_C, e : C \rightrightarrows C$ [Bor94, Prop. 6.5.4]. Thus, the requirement that a category be Cauchy complete is weaker than requiring the existence of all equalizers or all coequalizers.

Lemma 4.6 ([Bor94, Lem. 6.5.6])**.** Consider a category $\mathcal{C}$, and a representable functor $\mathcal{C}(-,C) : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$ such that all idempotents $C \to C$ split. Then any retract of $\mathcal{C}(-,C)$ is itself representable.

In particular, if $\mathcal{C}$ is Cauchy complete, then any retract of any representable functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$ is representable.

Proof. Consider such a $\mathcal{C}(-,C)$, and consider a retract of it as below.

$$F \xrightarrow{\iota} \mathcal{C}(-,C) \xrightarrow{\rho} F$$

Then $\iota\rho$ is an idempotent $\mathcal{C}(-,C) \to \mathcal{C}(-,C)$. Since the Yoneda embedding is full and faithful, we have $\iota\rho = \mathcal{C}(-,e)$ for some idempotent $e : C \to C$. But since this idempotent splits, we obtain a retraction of C

$$R \xrightarrow{i} C \xrightarrow{r} R$$

such that $ir = e$, and this produces a retraction of $\mathcal{C}(-,C)$.

$$\mathcal{C}(-,R) \xrightarrow{\mathcal{C}(-,i)} \mathcal{C}(-,C) \xrightarrow{\mathcal{C}(-,r)} \mathcal{C}(-,R)$$

But splittings of idempotents are unique by Lemma 4.5. Thus, $\mathcal{C}(-,R) \cong F$, and we conclude that F is representable. □

4.2. **Categories of display maps.**

Proposition 4.7. *Consider a Cauchy complete category $\mathcal{C}$. If $(\mathcal{C},\mathcal{D})$ is a category of display maps, then $(\mathcal{C},\overline{\mathcal{D}})$ is as well.*

Proof. Since $\mathcal{D} \subseteq \overline{\mathcal{D}}$ and $\mathcal{D}$ contains all isomorphisms and morphisms to the terminal object, then $\overline{\mathcal{D}}$ does as well. Since $\overline{\mathcal{D}}$ is the right class of a lifting pair, it is stable under pullback (Lemma 3.3). It only remains to show that pullbacks of morphisms of $\overline{\mathcal{D}}$ exist.

Consider a morphism $d : X \to Y$ of $\overline{\mathcal{D}}$ and a morphism $\alpha : A \to Y$ of $\mathcal{C}$. By Proposition 3.4, d is a retract in $\mathcal{C}/Y$ of some $d' : X' \to Y$ in $\mathcal{D}$. Let P denote the pullback diagram category, and let $D, D' : P \to \mathcal{C}$ denote the following two pullback diagrams in $\mathcal{C}$.

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle d} \\ A & \xrightarrow{\alpha} & Y \end{array} \qquad\qquad \begin{array}{ccc} & & X' \\ & & \downarrow{\scriptstyle d'} \\ A & \xrightarrow{\alpha} & Y \end{array}$$

Let c denote the functor $\mathcal{C} \to [P,\mathcal{C}]$ which sends an object m of $\mathcal{C}$ to the constant functor $c_m : P \to \mathcal{C}$ at m.

Then since d is a retract of d' in $\mathcal{C}/Y$, the functor D is a retract of D' in $[P,\mathcal{C}]$, and thus the functor $\mathrm{Nat}(c(-),D) : \mathcal{C} \to \mathcal{S}et$ is a retract of $\mathrm{Nat}(c(-),D') : \mathcal{C} \to \mathcal{S}et$. Now since we assume that there is a limit of the pullback diagram D', the functor $\mathrm{Nat}(c(-),D')$ is representable. Therefore, by Lemma 4.6, the functor $\mathrm{Nat}(c(-),D)$ is also representable, and we conclude that D has a limit.

Therefore, assuming that pullbacks of morphisms of $\mathcal{D}$ exist, pullbacks of morphisms of $\overline{\mathcal{D}}$ exist. □

4.3. Σ **types.** Since $\overline{\mathcal{D}}$ is closed under composition (Lemma 3.3), we immediately find that $(\mathcal{C}, \overline{\mathcal{D}})$ models Σ types. Note that for this result, we only use the hypothesis that $\mathcal{C}$ is Cauchy complete to ensure, by Proposition 4.7, that $(\mathcal{C}, \overline{\mathcal{D}})$ is a category of display maps.

Proposition 4.8. *Consider a Cauchy complete category* $\mathcal{C}$ *and a category of display maps* $(\mathcal{C}, \mathcal{D})$ *which models* Σ *and* `Id` *types.*

Then $(\mathcal{C}, \overline{\mathcal{D}})$ *is a category of display maps which models* Σ *types.*

Proof. $\overline{\mathcal{D}}$ is closed under composition by Lemma 3.3, and this means that $(\mathcal{C}, \overline{\mathcal{D}})$ models Σ types. □

4.4. `Id` **types.** There are two separate proofs that $(\mathcal{C}, \overline{\mathcal{D}})$ models `Id` types.

The first requires machinery that will be developed in subsequent papers of this series, so we cannot give the proof here. The proof can already be found in [Nor17], and so we include the statement here.

Note that the proof of the following statement does not use the hypothesis that $\mathcal{C}$ is Cauchy complete except to ensure that $(\mathcal{C}, \overline{\mathcal{D}})$ is a display map category by Proposition 4.7.

Proposition 4.9. *Consider a Cauchy complete category* $\mathcal{C}$ *and a category of display maps* $(\mathcal{C}, \mathcal{D})$ *which models* Σ *and* `Id` *types.*

Then $(\mathcal{C}, \overline{\mathcal{D}})$ *models* `Id` *types.*

Proof. Note that since $(\mathcal{C}, \mathcal{D})$ models `Id` types on objects, so does $(\mathcal{C}, \overline{\mathcal{D}})$. Then by Proposition A.1.5 of [Nor17], $(\mathcal{C}, \overline{\mathcal{D}})$ models `Id` types. □

We can, however, already present a proof of a similar result which is only impoverished by the additional hypothesis that the `Id` types are given functorially.

Definition 4.10. Consider a display map category $(\mathcal{C}, \mathcal{D})$ which models Σ.

For any object Y of $\mathcal{C}$, let $\{\mathcal{D}, \mathcal{C}\}_Y$ denote the full subcategory of the slice category $\mathcal{C}/Y$ spanned by those objects which are in $\mathcal{D}$. Also let $\mathbb{3}$ denote the poset containing the natural numbers $0, 1, 2$.

Say that $(\mathcal{C}, \mathcal{D})$ has *functorial* `Id` *types* if for each object Y, there is a functor $I_Y : \{\mathcal{D}, \mathcal{C}\}_Y \to \{\mathcal{D}, \mathcal{C}\}_Y^{\mathbb{3}}$ whose object part returns an `Id` type for each object $f : X \to Y$ of $\{\mathcal{D}, \mathcal{C}\}_Y$.

$$
\begin{tikzcd}
X \arrow[d, "f"] \\
Y
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
X \arrow[r, "r_f"] \arrow[dr, "f"'] & \mathtt{Id}(f) \arrow[r, "\epsilon_f"] \arrow[d, "\iota_f"] & X \times_Y X \arrow[dl, "f \times f"] \\
& Y &
\end{tikzcd}
$$

Proposition 4.11. *Consider a Cauchy complete category* $\mathcal{C}$*. Suppose that* $(\mathcal{C}, \mathcal{D})$ *is a category of display maps which models* Σ *types and functorial* `Id` *types. Then* $(\mathcal{C}, \overline{\mathcal{D}})$ *is a category of display maps which models functorial* `Id` *types.*

Note that in the following proof, we do use the hypothesis that $\mathcal{C}$ is Cauchy complete in an essential way.

Proof. Fix a slice $\mathcal{C}/Y$ and an object $e \in \overline{\mathcal{D}}$ in this slice. We want to construct an `Id` type on e. There is a $d \in \mathcal{D}$ such that e is a retract of d (Proposition 3.4). Since we have an `Id` type on d, we have the following diagram in $\mathcal{C}/Y$ (where i, s form the retraction and r_d, ϵ_d form the `Id` type on d).

$$
\begin{tikzcd}
e \arrow[d, "i"] & & e \times e \arrow[d, "i \times i"] \\
d \arrow[r, "r_d"] \arrow[d, "s"] & \iota_d \arrow[r, "\epsilon_d"] & d \times d \arrow[d, "s \times s"] \\
e & & e \times e
\end{tikzcd} \tag{$*$}
$$

The factorization gives a morphism $\iota_{\langle is, is \times is \rangle} : \iota_d \to \iota_d$ making the following diagram commute.

$$
\begin{tikzcd}
d \arrow[r, "r_d"] \arrow[d, "s"] & \iota_d \arrow[r, "\epsilon_d"] \arrow[dd, "\iota_{\langle is, is \times is \rangle}"] & d \times d \arrow[d, "s \times s"] \\
e \arrow[d, "i"] & & e \times e \arrow[d, "i \times i"] \\
d \arrow[r, "r_d"] & \iota_d \arrow[r, "\epsilon_d"] & d \times d
\end{tikzcd}
$$

Since $\langle is, is \times is \rangle$ is an idempotent and this factorization is given functorially, the morphism $\iota_{\langle is, is \times is \rangle}$ is also an idempotent. By Lemma 4.2, $\mathcal{C}/Y$ is Cauchy complete, so we can split the idempotent $\iota_{\langle is, is \times is \rangle}$. Then by Lemma 4.3, this extends to splittings of the rectangles in diagram ($*$) above. This gives us the following commutative diagram.

$$
\begin{tikzcd}
e \arrow[r, "r_e"] \arrow[d, "i"] & \iota_e \arrow[r, "\epsilon_e"] \arrow[d, "\iota_i"] & e \times e \arrow[d, "i \times i"] \\
d \arrow[r, "r_d"] \arrow[d, "s"] & \iota_d \arrow[r, "\epsilon_d"] \arrow[d, "\iota_r"] & d \times d \arrow[d, "s \times s"] \\
e \arrow[r, "r_e"] & \iota_e \arrow[r, "\epsilon_e"] & e \times e
\end{tikzcd}
$$

Now we see that the morphism ϵ_e is in $\overline{\mathcal{D}}_Y$ since it is a retract of $\epsilon_d \in \mathcal{D}_Y$.

Now, we need to show that for any $\alpha : a \to e$, the pullback $\alpha^* r_e$ is in ${}^{\boxslash}\overline{\mathcal{D}}_Y$. Let ϵ_{xi} denote the composition $\pi_i \epsilon_x$ for $x = d, e$ and $i = 0, 1$. Since r_e is a retract of

r_d, as shown in the following diagram, $\alpha^* r_e$ is a retract of $\alpha^* r_d$.

$$\begin{tikzcd}
 & & a \arrow[r, "\alpha^* r_e"] \arrow[dr, equal] & \alpha^*\iota_e \arrow[d, "\alpha^*\epsilon_{ei}"] & & & & e \arrow[r, "r_e"] \arrow[dr, equal] & \iota_e \arrow[d, "\epsilon_{ei}"] \\
 & \alpha^* d \arrow[ur, "\alpha^* s"] \arrow[r, "\alpha^* r_d"] \arrow[dr, equal] & \alpha^*\iota_d \arrow[ur, "\alpha^*\iota_s"] \arrow[d, "\alpha^*\epsilon_{di}"] & a \arrow[dddl, equal] & & & d \arrow[ur, "s"] \arrow[r, "r_d"] \arrow[dr, equal] & \iota_d \arrow[ur, "\iota_s"] \arrow[d, "\epsilon_{di}"] & e \arrow[dddl, equal] \\
a \arrow[ur, "\alpha^* i"] \arrow[r, "\alpha^* r_e"] \arrow[dr, equal] & \alpha^*\iota_e \arrow[ur, "\alpha^*\iota_i"] \arrow[d, "\alpha^*\epsilon_{ei}"] & \alpha^* d \arrow[ur, "\alpha^* s"] \arrow[dd, "\alpha^* s"] & & & e \arrow[ur, "i"] \arrow[r, "r_e"] \arrow[dr, equal] & \iota_e \arrow[ur, "\iota_i"] \arrow[d, "\epsilon_{ei}"] & d \arrow[ur, "s"] \arrow[dd, "s"] & \\
 & a \arrow[ur, "\alpha^* i"] \arrow[dr, equal] & & & & & e \arrow[ur, "i"] \arrow[dr, equal] & & \\
 & & a \arrow[rrrrr, "\alpha"] & & & & & e &
\end{tikzcd}$$

Since $\alpha^* r_d$ is in ${}^{\boxslash}\mathcal{D}_Y$ by hypothesis, and ${}^{\boxslash}\mathcal{D}$ is closed under retracts, we find that $\alpha^* r_e$ is in ${}^{\boxslash}\overline{\mathcal{D}}$.

Therefore, we have $\mathtt{Id}$ types for $(\mathcal{C}, \overline{D})$. $\square$

4.5. Π types.

Proposition 4.12. *Consider a Cauchy complete category of display maps $(\mathcal{C}, \mathcal{D})$ which models Σ types,* $\mathtt{Id}$ *types, and Π types. Then the category of display maps $(\mathcal{C}, \overline{\mathcal{D}})$ also models Π types.*

Proof. Consider morphisms $f : X \to Y$ and $g : W \to X$ which are both in $\overline{\mathcal{D}}$. We aim to obtain a Π type $\Pi_f g$.

Note that because

$$\rho(g) \times_Y \mathtt{Id}(Y) : (W \times_X \mathtt{Id}(X)) \times_Y \mathtt{Id}(Y) \to X \times_Y \mathtt{Id}(Y)$$

is a pullback of $\rho(g)$, it is in $\mathcal{D}$.

$$\begin{tikzcd}
(W \times_X \mathtt{Id}(X)) \times_Y \mathtt{Id}(Y) \arrow[r] \arrow[d, "\rho(g)\times_Y \mathtt{Id}(Y)"'] \arrow[dr, phantom, "\lrcorner", very near start] & W \times_X \mathtt{Id}(X) \arrow[d, "\rho(g)"] \\
X \times_Y \mathtt{Id}(Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
\mathtt{Id}(Y) \arrow[r, "\epsilon_0"] & Y
\end{tikzcd}$$

We will denote $\rho(g) \times_Y \mathtt{Id}(Y)$ as

$$M(\rho g) : M(f \circ \rho g) \to Mf$$

when it improves readability. (Note that the domain and codomain are indeed the middle objects of the factorizations of $f \circ \rho g$ and f, respectively.)

Since $M(\rho g)$ and ρf are in $\mathcal{D}$, we can form the Π type $\Pi_{\rho f} M(\rho g)$ with the following bijection for any $y : A \to Y$ in $\mathcal{C}$.

$$\mathcal{C}/Y(y, \Pi_{\rho f} M(\rho g)) \cong \mathcal{C}/Mf(\rho f^* y, M(\rho g))$$

This means that $\Pi_{\rho(f)} M(\rho g)$ represents the functor

$$\mathcal{C}/Mf(\rho f^* -, M(\rho g)) : \mathcal{C}/Y \to \mathcal{S}et.$$

We now show that $\mathcal{C}/X(f^* -, g)$ is a retract of this functor, so by Lemma 4.6, it will itself be representable.

Let i denote the natural transformation

$$\mathcal{C}/X(f^* -, g) \to \mathcal{C}/Mf(\rho f^* -, M(\rho g))$$

which at a morphism $z : Z \to Y$ in $\mathcal{C}$, takes a morphism $m : f^* z \to g$ in $\mathcal{C}/X$

$$\begin{tikzcd}
X \times_Y Z \arrow[dr, "f^* z"] \arrow[dd, "m"'] & \\
& X \\
W \arrow[ur, "g"] &
\end{tikzcd}$$

to the following morphism in $\mathcal{C}/Mf$

$$\begin{tikzcd}
X \times_{\epsilon_0} \mathtt{Id}(Y)_{\epsilon_1} \times Z \arrow[d, "a \times 1_{\mathtt{Id}Y} \times 1_Z"'] \arrow[dddr, "\rho f^* z"] & \\
\mathtt{Id}(X)_{(f\epsilon_0 \times f\epsilon_1)} \times_{(\epsilon_0 \times \epsilon_1)} \mathtt{Id}(Y)_{\epsilon_1} \times Z \arrow[d, "1_{\mathtt{Id}X} \times 1_{\mathtt{Id}Y} \times m(\epsilon_{1X} \times 1_Z)"'] \arrow[ddr, "\epsilon_{X0} \times 1"] & \\
\mathtt{Id}(X)_{(f\epsilon_0 \times f\epsilon_1)} \times_{(\epsilon_0 \times \epsilon_1)} \mathtt{Id}(Y)_{\epsilon_1} \times W \arrow[d, "b \times 1_{\mathtt{Id}Y}"'] \arrow[dr, "\epsilon_{X0} \times 1"] & \\
W \times_{\epsilon_0} \mathtt{Id}(X)_{f\epsilon_1} \times_{\epsilon_0} \mathtt{Id}(Y) \arrow[r, "M(\rho g)"] & X \times_Y \mathtt{Id}(Y)
\end{tikzcd}$$

where a and b are given by solutions to the following lifting problems.

$$\begin{tikzcd}
X \arrow[r, "r"] \arrow[d, "\lambda(f)"'] & \mathtt{Id}(X) \arrow[d, "\epsilon_0 \times f\epsilon_1"] \\
X_f \times_{\epsilon_0} \mathtt{Id}(Y) \arrow[r, "1 \times \epsilon_1"] \arrow[ur, dashed, "a"] & X \times Y
\end{tikzcd}
\qquad
\begin{tikzcd}
W \arrow[r, "\lambda(g)"] \arrow[d, "rg \times 1"'] & W_g \times_{\epsilon_0} \mathtt{Id}(X) \arrow[d, "\rho(g)"] \\
\mathtt{Id}(X)_{\epsilon_1} \times_g W \arrow[r, "\epsilon_0"] \arrow[ur, dashed, "b"] & X
\end{tikzcd}$$

(The morphism $\epsilon_0 \times f\epsilon_1$ is in $\overline{\mathcal{D}}$ because it is the composition of $\epsilon_0 \times \epsilon_1 : \mathtt{Id}(X) \to X \times X$ with $1 \times f : X \times X \to X \times Y$. The morphism $rg \times 1$ is in ${}^{\boxslash}\mathcal{D}$ because it is one of the pullbacks of $r : X \to \mathtt{Id}(X)$ ensured to be in ${}^{\boxslash}\mathcal{D}$ by the definition of $\mathtt{Id}$ types.)

Then let r denote the natural transformation

$$\mathcal{C}/Mf(\rho f^* -, M(\rho g)) \to \mathcal{C}/X(f^* -, g)$$

which at a morphism $z : Z \to Y$ in $\mathcal{M}$, takes a morphism $n : \rho f^* z \to M(\rho g)$ in $\mathcal{C}/Mf$

$$\begin{tikzcd}
X \times_{\epsilon_0} \mathtt{Id}(Y)_{\epsilon_1} \times Z \arrow[d, "n"] \arrow[dr, "\rho f^* z"] & \\
 & X \times_{\epsilon_0} \mathtt{Id}(Y) \\
W \times_{\epsilon_0} \mathtt{Id}(X)_{\epsilon_1} \times_{\epsilon_0} \mathtt{Id}(Y) \arrow[ur, "M(\rho g)"] &
\end{tikzcd}$$

to the following composition in $\mathcal{C}/X$

$$\begin{tikzcd}
X \times_Y Z \arrow[d, "1_X \times r_Y \times 1_Z"'] \arrow[ddrr, "f^* z"] & & \\
X \times_{\epsilon_0} \mathtt{Id}(Y)_{\epsilon_1} \times Z \arrow[d, "n"'] \arrow[drr, "\pi_X"] & & \\
W \times_{\epsilon_0} \mathtt{Id}(X)_{\epsilon_1} \times_{\epsilon_0} \mathtt{Id}(Y) \arrow[rr, "\epsilon_{X1}"] \arrow[d, "c"'] & & X \\
W \arrow[urr, "g"] & &
\end{tikzcd}$$

where c is a solution to the following lifting problem.

$$\begin{tikzcd}
W \arrow[r, equal] \arrow[d, "\lambda(g)"] & W \arrow[d, "g"] \\
W \times_{\epsilon_0} \mathtt{Id}(X) \arrow[r, "\rho(g)"] \arrow[ur, dashed, "c"] & X
\end{tikzcd}$$

Now we claim that

$$\mathcal{C}/X(f^* -, g) \xrightarrow{i} \mathcal{C}/Mf(\rho(f)^* -, M(\rho g)) \xrightarrow{r} \mathcal{C}/X(f^* -, g)$$

is a retract diagram. To that end, consider a morphism m of $\mathcal{C}/X(f^*z, g)$. Then $ri(m)$ is the following composition.

$$\begin{tikzcd}
X \times_Y Z \arrow[d, "1_X \times r_Y \times 1_Z"'] \arrow[dddr, "f^*z"] & \\
X \times_{\epsilon_0} \mathtt{Id}(Y)_{\epsilon_1} \times Z \arrow[d, "a \times 1_{\mathtt{Id}Y} \times 1_Z"'] \arrow[ddr, "\pi_X"] & \\
\mathtt{Id}(X)_{(f\epsilon_0 \times f\epsilon_1)} \times_{(\epsilon_0 \times \epsilon_1)} \mathtt{Id}(Y)_{\epsilon_1} \times Z \arrow[d, "1_{\mathtt{Id}X} \times 1_{\mathtt{Id}Y} \times m(\epsilon_{1X} \times 1_Z)"'] \arrow[dr, "\epsilon_X 1"] & \\
\mathtt{Id}(X)_{(f\epsilon_0 \times f\epsilon_1)} \times_{(\epsilon_0 \times \epsilon_1)} \mathtt{Id}(Y)_{\epsilon_1} \times W \arrow[d, "b \times 1_{\mathtt{Id}Y}"'] \arrow[r, "\epsilon_X 1"] & X \\
W \times_{\epsilon_0} \mathtt{Id}(X)_{f\epsilon_1} \times_{\epsilon_0} \mathtt{Id}(Y) \arrow[d, "c"'] \arrow[ur, "\epsilon_X 1"] & \\
W \arrow[uur, "g"'] &
\end{tikzcd}$$

The composition $a \circ (1_X \times r_Y) : X \to \mathtt{Id}(X)$ is r_X. Thus, the composite of the first three vertical morphisms in the above diagram is

$$r_X \times r_Y \times m : X \times_Y Z \to \mathtt{Id}(X)_{(f\epsilon_0 \times f\epsilon_1)} \times_{(\epsilon_0 \times \epsilon_1)} \mathtt{Id}(Y)_{\epsilon_1} \times W.$$

Moreover, $b \circ (r_X \times 1_W) : W \to W \times_{\epsilon_0} \mathtt{Id}(X)$ is $1_W \times r_X$ so the composite of the first four morphisms above is

$$m \times r_X \times r_Y : X \times_Y Z \to W \times_{\epsilon_0} \mathtt{Id}(X)_{f\epsilon_1} \times_{\epsilon_0} \mathtt{Id}(Y).$$

The composite $c \circ (1_W \times r_X) : W \to W$ is the identity, so the vertical composite above is m. Therefore, $ri(m) = m$, and i and r form a retract.

Now by Lemma 4.6, we can conclude that $\mathcal{C}/X(f^*-, g) : \mathcal{C}/Y \to \mathcal{S}et$ is representable by an object which we will denote by $\Pi_f g$. Furthermore, $\Pi_f g$ is a retract of $\Pi_{\rho f} M(\rho g)$. Since $\Pi_{\rho f} M(\rho g)$ is in $\mathcal{D}$, we can conclude that $\Pi_f g$ is in $\overline{\mathcal{D}}$, the retract closure of $\mathcal{D}$. Therefore, $(\mathcal{C}, \overline{\mathcal{D}})$ does in fact model Π types. $\square$

4.6. **Summary.** Putting together Propositions 4.7, 4.8, 4.9, and 4.12, we get the following theorem.

Theorem 4.13. *Consider a Cauchy complete category $\mathcal{C}$ and a display map category $(\mathcal{C}, \mathcal{D})$ which models Σ types and* `Id` *types. Then $(\mathcal{C}, \overline{\mathcal{D}})$ is again a display map category modeling Σ and* `Id` *types.*

Furthermore, if $(\mathcal{C}, \mathcal{D})$ additionally models Π types, then $(\mathcal{C}, \overline{\mathcal{D}})$ also models Π types.

Proof. By Proposition 4.7, $(\mathcal{C}, \overline{\mathcal{D}})$ is a category with display maps. By Proposition 4.8, it models Σ types. By Proposition 4.9, it models `Id` types. By Proposition 4.12, it models Π types. $\square$

5. Display map categories reflected in weak factorization systems.

In this last section, we remark that our main theorem, 4.13, can be phrased more categorically as Theorem 5.1 below.

Let $\mathcal{C}$ be a Cauchy complete category. Then let $\mathcal{S}(\mathcal{C})$ denote the category whose objects are subclasses $\mathcal{M}$ of morphisms of $\mathcal{C}$ and whose morphisms $\mathcal{M} \to \mathcal{N}$ are inclusions $\mathcal{M} \subseteq \mathcal{N}$. Then we can identify the following four full subcategories of $\mathcal{S}(\mathcal{C})$:

- $\mathsf{DMC}_{\Sigma,\mathtt{Id}}(\mathcal{C})$, the full subcategory of $\mathcal{S}(\mathcal{C})$ spanned by those $\mathcal{M}$ such that $(\mathcal{C},\mathcal{M})$ is a display map category with Σ and `Id` types;
- $\mathsf{DMC}_{\Sigma,\mathtt{Id},\Pi}(\mathcal{C})$, the full subcategory of $\mathcal{S}(\mathcal{C})$ spanned by those $\mathcal{M}$ such that $(\mathcal{C},\mathcal{M})$ is a display map category with Σ, `Id`, and Π types;
- $\mathsf{WFS}_{\Sigma,\mathtt{Id}}(\mathcal{C})$, the full subcategory of $\mathsf{DMC}_{\Sigma,\mathtt{Id}}(\mathcal{C})$ spanned by those $\mathcal{M}$ such that $({}^{\boxslash}\mathcal{M},\mathcal{M})$ is a weak factorization system; and
- $\mathsf{WFS}_{\Sigma,\mathtt{Id},\Pi}(\mathcal{C})$, the full subcategory of $\mathsf{DMC}_{\Sigma,\mathtt{Id},\Pi}(\mathcal{C})$ spanned by those $\mathcal{M}$ such that $({}^{\boxslash}\mathcal{M},\mathcal{M})$ is a weak factorization system.

A morphism $\mathcal{M} \to \mathcal{N}$ in $\mathsf{DMC}_{\Sigma,\mathtt{Id}}(\mathcal{C})$ or $\mathsf{DMC}_{\Sigma,\mathtt{Id},\Pi}(\mathcal{C})$ preserves Σ types and Π types if they exist. That is, for composable $f, g \in \mathcal{M}$, since the Σ types $\Sigma_f^{\mathcal{M}} g$ in $(\mathcal{C},\mathcal{M})$ and $\Sigma_f^{\mathcal{N}} g$ in $(\mathcal{C},\mathcal{N})$ are both defined to be the composition $f \circ g$, we have $\Sigma_f^{\mathcal{M}} g = \Sigma_f^{\mathcal{N}} g$. Since the Π types $\Pi_f^{\mathcal{M}} g$ in $(\mathcal{C},\mathcal{M})$ and $\Pi_f^{\mathcal{N}} g$ in $(\mathcal{C},\mathcal{N})$ are both defined with the same universal property, we have $\Pi_f^{\mathcal{M}} g \cong \Pi_f^{\mathcal{N}} g$.

A morphism $\mathcal{M} \to \mathcal{N}$ in $\mathsf{DMC}_{\Sigma,\mathtt{Id}}(\mathcal{C})$ or $\mathsf{DMC}_{\Sigma,\mathtt{Id},\Pi}(\mathcal{C})$ does not however preserve `Id` types. There is rather a comparison morphism $c_f : \mathtt{Id}^{\mathcal{N}}(f) \to \mathtt{Id}^{\mathcal{M}}(f)$ for any $f : X \to Y$ in $\mathcal{M}$ induced by the following lifting problem in $\mathcal{C}/Y$.

$$\begin{tikzcd} f \arrow[r, "r^{\mathcal{M}}"] \arrow[d, "r^{\mathcal{N}}"'] & \mathtt{Id}^{\mathcal{M}}(f) \arrow[d, "\epsilon^{\mathcal{M}}"] \\ \mathtt{Id}^{\mathcal{N}}(f) \arrow[r, "\epsilon^{\mathcal{N}}"'] \arrow[ur, dashed, "c_f"] & f \times f \end{tikzcd}$$

The lift c_f exists since $r^{\mathcal{N}} \in {}^{\boxslash}\mathcal{N}$ and $\epsilon^{\mathcal{M}} \in \mathcal{M} \subseteq \mathcal{N}$.

One might want to consider a category of display map categories with the structure, and not merely the existence, of identity type and morphisms between them which preserve this structure. However, we give here a simplified version of this story to just expose this reflective relationship without being encumbered by technicalities.

Now we can state our main theorem as the existence of a reflector.

Theorem 5.1. *Consider a Cauchy complete category $\mathcal{C}$. The category $\mathsf{WFS}_{\Sigma,\mathtt{Id}}(\mathcal{C})$ is a reflective subcategory of $\mathsf{DMC}_{\Sigma,\mathtt{Id}}(\mathcal{C})$, and $\mathsf{WFS}_{\Sigma,\mathtt{Id},\Pi}(\mathcal{C})$ is a reflective subcategory of $\mathsf{DMC}_{\Sigma,\mathtt{Id},\Pi}(\mathcal{C})$. That is, there are left adjoints L in the diagram below.*

$$\begin{tikzcd} \mathsf{WFS}_{\Sigma,\mathtt{Id}}(\mathcal{C}) \arrow[r, hook, shift right] \arrow[r, phantom, "\perp"] & \mathsf{DMC}_{\Sigma,\mathtt{Id}}(\mathcal{C}) \arrow[l, shift right, "L"'] \\ \mathsf{WFS}_{\Sigma,\mathtt{Id},\Pi}(\mathcal{C}) \arrow[u, hook] \arrow[r, hook, shift right] \arrow[r, phantom, "\perp"] & \mathsf{DMC}_{\Sigma,\mathtt{Id},\Pi}(\mathcal{C}) \arrow[l, shift right, "L"'] \arrow[u, hook] \end{tikzcd}$$

Proof. Consider the endofunctor on $\mathcal{S}(\mathcal{C})$ given on objects by $L(\mathcal{M}) := \overline{\mathcal{M}}$. By Theorem 4.13, this can be restricted to functors

$$L : \mathsf{DMC}_{\Sigma,\mathtt{Id}}(\mathcal{C}) \to \mathsf{WFS}_{\Sigma,\mathtt{Id}}(\mathcal{C}),$$
$$L : \mathsf{DMC}_{\Sigma,\mathtt{Id},\Pi}(\mathcal{C}) \to \mathsf{WFS}_{\Sigma,\mathtt{Id},\Pi}(\mathcal{C}).$$

To see that these are the left adjoints shown in the statement, we need to show for any $\mathcal{D}$ in $\mathsf{DMC}_{\Sigma,\mathtt{Id}}(\mathcal{C})$ and any $\mathcal{R}$ in $\mathsf{WFS}_{\Sigma,\mathtt{Id}}(\mathcal{C})$ that

$$\hom(L\mathcal{D}, \mathcal{R}) \cong \hom(\mathcal{D}, \mathcal{R}),$$

or, equivalently, that there is an inclusion $\overline{\mathcal{D}} \subseteq \mathcal{R}$ if and only if there is an inclusion $\mathcal{D} \subseteq \mathcal{R}$.

If $\overline{\mathcal{D}} \subseteq \mathcal{R}$, then since $\mathcal{D} \subseteq \overline{\mathcal{D}}$, we have that $\mathcal{D} \subseteq \mathcal{R}$.

If $\mathcal{D} \subseteq \mathcal{R}$, then $\overline{\mathcal{D}} \subseteq \overline{\mathcal{R}}$. Since $\overline{\mathcal{R}} = \mathcal{R}$ by [MP12, Proposition 14.1.8], we have that $\overline{\mathcal{D}} \subseteq \mathcal{R}$. □

6. Future work

In this paper, we have described a relationship between display map categories and weak factorization systems. We hope to upgrade this in the future to a description of the relationship between comprehension categories and more structured weak factorization systems. In particular, the perspective taken in Theorem 5.1 is perhaps a bit extravagant, given that it describes a relationship between pre-ordered sets, already described in Theorem 4.13, in fancy categorical language. However, this more categorical result will be the one appropriate for strengthening in a study of more structured notions of categorical interpretations of type theory and weak factorization systems.

References

[BG12] B. van den Berg and R. Garner. "Topological and simplicial models of identity types". In: *ACM Trans. Comput. Log.* 13.1 (2012), 3:1–3:44.

[Bor94] F. Borceux. *Handbook of categorical algebra.* Vol. 1. Cambridge, UK: Cambridge University Press, 1994.

[GG08] N. Gambino and R. Garner. "The identity type weak factorisation system". In: *Theoret. Comput. Sci.* 409.1 (2008), pp. 94–109.

[Jac93] B. Jacobs. "Comprehension categories and the semantics of type dependency". In: *Theoret. Comput. Sci.* 107.2 (1993), pp. 169–207.

[Joy13] A. Joyal. *A categorical approach to homotopy type theory.* Talk given at Centre de Recerca Matemàtica, 2013. URL: `http://www.crm.cat/en/Activities/Documents/joyal-crm-2013.pdf`.

[LW15] P. L. Lumsdaine and M. A. Warren. "The local universes model: an overlooked coherence construction for dependent type theories". In: *ACM Trans. Comput. Log.* 16.3 (2015), Art. 23, 31.

[MP12] J. P. May and K. Ponto. *More concise algebraic topology.* Chicago: University of Chicago Press, 2012.

[Nor17] P. R. North. "Type theoretic weak factorization systems". PhD thesis. University of Cambridge, 2017.

[Str93] T. Streicher. "Investigations into intensional type theory". Habilitationsschrift. Ludwig Maximilian University of Munich, 1993.

[Tay99] P. Taylor. *Practical foundations of mathematics*. Cambridge, UK: Cambridge University Press, 1999.

[Emm14] J. Emmenegger. *A category-theoretic version of the identity type weak factorization system*. 2014. arXiv: `1412.0153 [math.LO]`.

[Uni13] The Univalent Foundations Program. *Homotopy type theory: univalent foundations of mathematics*. Institute for Advanced Study: `http://homotopytypetheory.org/book`, 2013.