The enriched Curry-Howard-Lambek correspondence

Abstract—**In this work, we embark on a program to develop type theory with semantics in enriched categories. Category theory has gained a lot of flexibility and expressive power by enlarging its domain of study to include enriched categories (which include higher categories) – these are like categories except that the homs live not in the category of sets but in some specified base category. On the other hand, the Curry-Howard-Lambek correspondence is the basis for much of modern type theory and categorical logic: it connects logic to the simply typed lambda calculus which then is seen as the internal language for cartesian closed categories. Here, we extend the Curry-Howard-Lambek correspondence to develop an enriched simply typed lambda calculus which we show is an internal language for enriched cartesian closed categories, and we exhibit a corresponding logic. We are guided by intuition from our most basic nontrivial example – fuzzy logic and set theory – and we argue that the logic that we obtain in this case is a compelling alternative to the intuitionistic fuzzy logics that exist in the literature.**

I. Introduction

A. Motivation

In this work, we execute the first step in a program to develop type theory with semantics in enriched categories [16].

Motivation from category theory: An enriched category has, like an ordinary category, a collection of objects, but its homs are not sets but rather objects in some other category, called the base of enrichment. Many mathematical objects naturally form an enriched category: for example, the collection of homomorphisms between two vector spaces could be considered as a set, but it is much more useful to recognize that it forms a vector space itself.

It is difficult to overstate the importance of enriched category theory within category theory itself. Indeed, to study categories themselves, one needs to work in the 2-category – that is, category-enriched category – of categories. Many approaches to higher category theory use enriched category theory (and the other approaches use closely related techniques). Classically, topologically enriched categories, simplicial categories, dg-categories, linear categories, abelian categories, etc. are all important objects of study in algebraic geometry and topology and are enriched categories; preordered sets and metric spaces can also be seen as enriched categories.

Many of these are directly relevant to theoretical computer science as the semantics for homotopy, cubical, directed, and simplicial type theories. One needs only look at recent publications in category theory to see other relevant applications of enriched category theory: a selection of such papers from the last year includes applications in synthetic domain theory [21], categorical probability theory [17, 24], linear logic [25], differential categories [18], quantum programming languages [28], reversible programming languages [6], and information dynamics [11]. Thus, developing an internal language for enriched categories that could form the basis of a computer proof assistant is a well-motivated problem.

Motivation from computer science: Besides the applications of enriched category theory in computer science above, we see two main motivations from computer science.

First, we think that giving enriched categorical semantics can clarify some features of functional programming languages. When understanding what a type theory or functional programming language 'means' – in either a technical or intuitive sense – there is a (very natural) tendency to conflate the collection of the terms of a type T with the type T itself. Under usual categorical semantics, this corresponds to conflating the set of morphisms $* \to T$ with the type T itself (or in a dependently typed setting, sections of $\Gamma.T \to \Gamma$ with the type T in context Γ). Thus when working with, for instance, the simply typed lambda calculus, one can feel that one is working directly with sets, even though they are just working with the internal language of (cartesian closed) categories *enriched* in sets. We concede that this feeling is justified by the fact that $\mathcal{S}et$ is itself such a category, but consider for instance the polymorphic term `undefined` of Haskell. Justified by semantics of related languages in categories such as dcpos, this feature has been added into Haskell to, roughly, make the types of Haskell behave more like dcpos. But to be more precise, by doing this we are not making the types themselves more like dcpos so much as the sets of terms of each type. Actually, it is appropriate to not view the collections of terms as sets anymore, but rather as pointed sets. Thus, from this perspective it is more appropriate to take semantics of a language with a polymorphic `undefined` term in pointed-set-enriched categories rather than in (set-enriched) categories, interpreting `undefined` as the point in each set. We are sure that the difference between types and their sets of elements is well understood by most users of type theories and functional programming languages, but we think for the reason described here that an enriched categorical perspective on semantics of programming languages is a natural one. See Examples 4, 7, 12.

Second, we see the development of a mathematical theory of the modal and similar (e.g., quantitative, directed) type theories that are currently proliferating (see e.g. [23, 19, 4, 22, 13, 14]) as one of the most important problems in current type theory research. We see the core contribution of such type theories to be the inclusion of annotations in the syntax that govern how terms can be used. In this work, we also enrich the simply typed lambda calculus with term annotations, but in a different way than in the aforementioned works (only to the right of the turnstile) and with different semantics (in enriched categories). We thus see this work as illuminating yet another corner in the space of possibilities for type theories with such annotations. See Example 17 for further discussion.

Motivation from logic: This work started as an attempt to understand a type theory with semantics in fuzzy sets [12], as fuzzy sets seemed to be a reasonably small first step away from usual set-based semantics to more generally enriched ones. In the end, the theory that we present here is much more general than the one for fuzzy sets, but this remains our favorite example. In particular, we present a new intuitionistic fuzzy logic with the advantage over others in the literature in that it comes with a Curry-Howard-Lambek correspondence.

B. Contributions

In this work, we present what we call the $\mathcal{W}$-enriched simply typed lambda calculus (Definition 3). This has sound (Theorem 1) and complete (Theorem 2) semantics in cartesian closed $\mathcal{V}$-categories (Definition 5) where $\mathcal{V}$ is a $\mathcal{W}$-relative monoidal category (Definition 4) and $\mathcal{W}$ is a monoidal category.

When $\mathcal{W}$ is a monoidal poset $\mathbb{W}$, we also present $\mathbb{W}$-natural deduction (Definition 6) with sound and complete semantics in cartesian closed $\mathbb{V}$-categories (Theorem 5) where $\mathbb{V}$ is a $\mathbb{W}$-relative monoidal poset. We show that this has a Curry-Howard correspondence with the $\mathbb{W}$-enriched simply typed lambda calculus (Theorem 4).

When $\mathcal{W}$ and $\mathbb{W}$ are the trivial monoidal category, our definitions and results reduce to the classical case: we obtain the simply typed lambda calculus with its semantics in cartesian closed categories and natural deduction with its Curry-Howard correspondence to the simply typed lambda calculus.

We give many examples of categories that fulfill the hypotheses of our results.

In particular, we analyze to what extent the syntax is sound and complete when we restrict the class of models to those relevant in the fuzzy case. That is, we show to what extent the $\mathbb{W}$-enriched simply typed lambda calculus has a sound and complete interpretation in cartesian closed $\mathcal{S}et(\mathbb{W})$-categories (Theorem 3) (where $\mathcal{S}et(\mathbb{W})$ is the category of fuzzy sets), and to what extent $\mathbb{W}$-natural deduction has a sound and complete interpretation in cartesian closed $\mathbb{W}_0$-categories (Theorem 6) (where $\mathbb{W}_0$ is $\mathbb{W}$ freely adjoined with an absorbing, bottom element). We argue that $\mathbb{W}$-natural deduction should be viewed as an intuitionistic fuzzy logic, the only one in the literature that is part of a Curry-Howard-Lambek correspondence.

C. Related work

We do not know of a type theory specifically designed to have semantics in enriched categories.

Fiore [9] gives an interpretation of FPC (a type theory with sums, products, exponentials, and recursive types) in $\mathcal{CPO}$-categories. We obtain as an instance of our results a simply typed calculus with semantics in $\mathcal{CPO}$-categories (Example 18), but we do not study recursion in this work.

A dependent type theory for bicategories was developed in [2]. Though bicategories are not exactly enriched categories, if one restricts to 2-categories, then that type theory could be seen as one for category-enriched categories. We do obtain as another instance of our results a calculus with semantics in 2-categories (Example 9), but our syntax is simply typed and theirs is dependently typed.

There are superficial, syntactic similarities with modal and quantitative type theories, as discussed above, and less superficial similarities as explained in Example 17. We hope to give a formal, semantic comparison of the enriched type theory that we develop with modal and quantitative type theories in future work.

Lastly, we give a careful comparison of our $\mathbb{W}$-natural deduction with the other intuitionistic fuzzy logics in the literature in Section IV-C.

II. Syntax

Our syntax starts with a monoidal poset (defined in the rules below) $\mathbb{W}$ of *weights*. If $\mathbb{W} = \mathbb{1}$, we obtain the simply typed lambda calculus [7]. Thus, we view this as a family of extensions of the simply typed lambda calculus, parametrized by $\mathbb{W}$, and call it the $\mathbb{W}$-*enriched simply typed lambda calculus*.

It is possible to generalize from a monoidal poset to a monoidal category $\mathbb{W}$, but we first give the rules in the less general case, since they are simpler. We give the alternate rules for a monoidal category in Section II-G below.

A. Basic rules pertaining to $\mathbb{W}$

We first fix a monoidal poset $\mathbb{W}$. *Monoidal poset* is defined formally by the rules at the end of this subsection, but it can alternately defined as a posetal, monoidal category. The first three judgments of our calculus will be the following.

$$w \text{ wt} \qquad w = w' \text{ wt} \qquad w \leq w' \text{ wt}$$

The first judgment w wt is read 'w is a weight' and holds when $w \in \mathbb{W}$. The second holds when $w = w'$, and the third holds when $w \leq w'$ (provided $w, w' \in \mathbb{W}$).

Instead of starting with a monoidal poset $\mathbb{W}$ external to the calculus, one could take the alternate approach $\mathbb{W}$ as the first layer of this calculus. That is, we could give the following rules, which constitute the definition of *monoidal* poset, together with constant symbols adding generators and axioms asserting relations. In this work, we will use the first approach (assuming an external monoidal poset $\mathbb{W}$) because our results hold for any such $\mathbb{W}$, not just finitely axiomatizable ones. However in an implementation, one might prefer to use the second approach.

$$\textsc{$\mathbb{W}$-Refl}\ \frac{w \text{ wt}}{w \leq w \text{ wt}} \qquad \textsc{$\mathbb{W}$-Trans}\ \frac{u \leq v \text{ wt} \qquad v \leq w \text{ wt}}{u \leq w \text{ wt}}$$

$$\textsc{$\mathbb{W}$-ASym}\ \frac{v \leq w \text{ wt} \qquad w \leq v \text{ wt}}{v = w \text{ wt}} \qquad \textsc{$\mathbb{W}$-Prod}\ \frac{v, w \text{ wt}}{vw \text{ wt}} \qquad \textsc{$\mathbb{W}$-Id}\ \frac{}{1 \text{ wt}}$$

$$\textsc{$\mathbb{W}$-IdL}\ \frac{w \text{ wt}}{1w = w \text{ wt}} \qquad \textsc{$\mathbb{W}$-IdR}\ \frac{w \text{ wt}}{w = w1 \text{ wt}} \qquad \textsc{$\mathbb{W}$-Ass}\ \frac{u, v, w \text{ wt}}{u(vw) = (uv)w \text{ wt}}$$

$$\textsc{$\mathbb{W}$-Func}\ \frac{t \leq u \text{ wt} \qquad v \leq w \text{ wt}}{tv \leq uq \text{ wt}}$$

B. Structural rules

We are defining a generalized simply typed lambda calculus, so in addition to the weight judgments we have the usual judgments (usual, except that the term judgment includes a weight). First, we have the usual judgments pertaining to types and contexts.

$$T \text{ type} \qquad T = T' \text{ type} \qquad \Gamma \text{ ctx} \qquad \Gamma = \Gamma' \text{ ctx}$$

Our term judgments now include a weight: for a context Γ, a type T, and a weight w, we have the following judgment (read as 't is a term with weight w of type T in context Γ') together with a corresponding equality judgment.

$$\Gamma \vdash t :_w T \qquad \Gamma \vdash t = t' :_w T$$

For the four equality judgments that we have introduced, we refrain from writing the rules that state that they are congruences, but we certainly assume them.

The context formation rules are exactly the same as in the usual simply typed lambda calculus.

$$\textsc{C-Emp}\ \frac{}{\Diamond \text{ ctx}} \qquad \textsc{C-Ext}\ \frac{\Gamma \text{ ctx} \qquad T \text{ type}}{\Gamma, x : T \text{ ctx}}$$

We have the following structural rules. Note that the variable (Var) and substitution (Subst) rules use the weights in a nontrivial way, unlike weakening (Wk).

$$\textsc{Var}\ \frac{\Gamma, x : T, \Delta \text{ ctx}}{\Gamma, x : T, \Delta \vdash x :_1 T} \qquad \textsc{Wk}\ \frac{\Gamma, \Delta \vdash t :_w T \qquad S \text{ type}}{\Gamma, s : S, \Delta \vdash t :_w T}$$

$$\textsc{Subst}\ \frac{\Gamma, x : S, \Delta \vdash t :_w T \qquad \Gamma \vdash s :_v S}{\Gamma, \Delta \vdash t[s/x] :_{vw} T}$$

We assume that this substitution, as an implicit substitution, satisfies the usual rules that usually taken to be self-evident: that $t[x/x] = t$, etc.

We also have a weakening rule for weights.

$$\textsc{$\mathbb{W}$-Wk}\ \frac{\Gamma \vdash t :_w T \qquad v \leq w \text{ wt}}{\Gamma \vdash t :_v T}$$

C. Intuition: fuzzy sets

We pause here to preview a bit of intuition coming from the semantics (see Section III). Whereas the semantics of the simply typed lambda calculus lie in (cartesian closed) categories – that is, categories enriched in sets – our semantics will lie in enriched categories, and the illustrating semantics will lie in categories enriched in *fuzzy* sets.

Definition 1: For a monoidal poset $\mathbb{W}$, a $\mathbb{W}$*-fuzzy set* (or simply, fuzzy set) is a pair (S, f) where S is a set and f is a function $S \to \mathbb{W}$ [12]. A *morphism* $\alpha : (S, f) \to (T, g)$ *of* $\mathbb{W}$*-fuzzy sets* is a function $\alpha : S \to T$ such that $f(x) \leq g\alpha(x)$ for all $x \in S$. We denote resulting category of $\mathbb{W}$-fuzzy sets by $\mathcal{S}et(\mathbb{W})$.

This category has a monoidal structure whose unit is $(*, 1)$, the singleton with the constant function at $1 \in \mathbb{W}$, and whose tensor is given by $(S, f) \otimes (T, g) := (S \times T, \lambda x. f(x)g(x))$. ◀

Example 1: $\mathcal{S}et(\mathbb{1})$ is isomorphic to $\mathcal{S}et$. $\mathcal{S}et(\mathbb{I})$ is the category of fuzzy sets in the sense of Zadeh [29]. ◀

Thus, categories enriched in $\mathbb{W}$-fuzzy sets are equivalently categories that have additional structure: each morphism is equipped with 'weight' in $\mathbb{W}$ (such that each identity morphism has weight 1 and the weight of $g \circ f$ is the product of the weights of f and g).

We will show in Theorem 1 that there is an interpretation of the following rules in categories enriched in $\mathbb{W}$-fuzzy sets.

There, a judgment $\Gamma \vdash t :_w T$ will be interpreted to mean that there is a morphism from Γ to T with weight *at least* w.

If we take $\mathbb{W}$ to be the trivial monoidal poset $\mathbb{1}$ (i.e., the singleton $\{1\}$ with its unique ordering and multiplication), we obtain exactly the simply typed lambda calculus with an extra, meaningless piece of syntax that can be erased: when a colon appears to the right of turnstile, it will appear not as ':' but as '$:_1$'.

If we take $\mathbb{W}$ to be the booleans $\mathbb{B}$ (i.e., $\{0, 1\}$ with the order corresponding to implication and multiplication corresponding to conjunction), we also obtain a calculus similar to the simply typed lambda calculus. The judgment $\Gamma \vdash t :_1 T$ is then interpreted as a morphism from Γ to T with weight 1, and a judgment $\Gamma \vdash t :_0 T$ is interpreted as a morphism Γ to T with any weight. Thus, the judgment $\Gamma \vdash t :_1 T$ carries more information than $\Gamma \vdash t :_0 T$. One might want to regard the former judgment as meaning that there is a morphism $\Gamma \to T$ and the latter as carrying no information; one could formalize this intuition in part by showing that our $\mathbb{B}$-enriched simply typed lambda calculus is conservative over the simply typed lambda calculus.

If we take the fixed monoidal poset to be the unit interval $\mathbb{I}$ (i.e., the subset $[0, 1]$ of the real numbers with the order and multiplication inherited from them; note the natural inclusion $\mathbb{B} \hookrightarrow \mathbb{I}$), then a judgment $\Gamma \vdash t :_w T$ is interpreted as a morphism from Γ to T of weight w. We might understand this as asserting with some non-binary confidence w that there is a morphism from Γ to T.

D. Rules for weightings

Going from category theory to enriched category theory, products generalize to weighted products. However, to take a weighted binary product of two objects in an enriched category, one can take 'weighted unary products' (which in the syntax we will call the *weighting of a type T by w*) of each object individually and then take an (unweighted) binary product. We axiomatize that approach here, first giving the weightings of types.

$$\textsc{Wtg-Form}\quad \frac{T \text{ type} \qquad w \text{ wt}}{T^w \text{ type}} \qquad\qquad \textsc{Wtg-Intro}\quad \frac{\Gamma \vdash t :_{vw} T \qquad w \text{ wt}}{\Gamma \vdash t^w :_v T^w}$$

$$\textsc{Wtg-Elim}\quad \frac{\Gamma \vdash t :_v T^w}{\Gamma \vdash t^{\backslash w/} :_{vw} T} \qquad\qquad \textsc{Wtg-}\beta\quad \frac{\Gamma \vdash t :_{vw} T \qquad w \text{ wt}}{\Gamma \vdash t = t^{w\backslash w/} :_{vw} T}$$

$$\textsc{Wtg-}\eta\quad \frac{\Gamma \vdash t :_v T^w}{\Gamma \vdash t = t^{\backslash w/w} :_v T^w}$$

Note in the above rules that $\backslash w/$ is not a new weight; rather $t^{\backslash w/}$ is a new term symbol. Also note that $t^{w\backslash w/}$ could have been more carefully written as $(t^w)^{\backslash w/}$, but we write it without parentheses for readability (similarly for $t^{\backslash w/w}$).

Example 2: One can observe that the rules governing T^1 above present a syntactic isomorphism between T and T^1. Thus, when $\mathbb{W} = \mathbb{1}$ and we compare this with the original simply typed lambda calculus, we find that the types are duplicated, but in a nonessential way.

If there is a zero (i.e., initial and absorbing) element $0 \in \mathbb{W}$ (e.g., in $\mathbb{B}$ and $\mathbb{I}$), then the terms of T^0 correspond to terms of T of weight 0.

For an arbitrary w, the terms of weight 1 in T^w correspond to terms of weight w in T, so we can understand T^w as a scaling of T by w. ◀

E. Binary products

Now, we give the rules for products of types. The rules that contain a parameter i stand for two rules: one for $i = 1$ and one for $i = 2$.

$$\times\textsc{-Form}\quad \frac{T_1 \text{ type} \qquad T_2 \text{ type}}{T_1 \times T_2 \text{ type}} \qquad\qquad \times\textsc{-Intro}\quad \frac{\Gamma \vdash t_1 :_w T_1 \qquad \Gamma \vdash t_2 :_w T_2}{\Gamma \vdash \langle t_1, t_2 \rangle :_w T_1 \times T_2}$$

$$\times\textsc{-Elim}\quad \frac{\Gamma \vdash p :_w T_1 \times T_2}{\Gamma \vdash \pi_i p :_w T_i} \qquad\qquad \times\text{-}\beta\quad \frac{\Gamma \vdash t_1 :_w T_1 \qquad \Gamma \vdash t_2 :_w T_2}{\Gamma \vdash \pi_i \langle t_1, t_2 \rangle = t_i :_w T_i}$$

$$\times\text{-}\eta\quad \frac{\Gamma \vdash p :_w T_1 \times T_2}{\Gamma \vdash \langle \pi_1 p, \pi_2 p \rangle = p :_w T_1 \times T_2}$$

Note that when $\mathbb{W} = \mathbb{1}$, these are the rules for binary products in the simply typed lambda calculus (with some erasable subscripts). More generally, for each weight w, the rules can be seen as describing usual products of 'subtypes' of terms of weight w. We might say that these products are taken 'weightwise'.

Observe that if $\mathbb{W}$ has binary meets $\wedge$, then using $\mathbb{W}$-Wk, the rules $\times$-Intro and $\times$-β are equivalent to the following rules.

$$\times\textsc{-Intro-Alt}\quad \frac{\Gamma \vdash t_1 :_{w_1} T_1 \qquad \Gamma \vdash t_2 :_{w_2} T_2}{\Gamma \vdash \langle t_1, t_2 \rangle :_{w_1 \wedge w_2} T_1 \times T_2}$$

$$\times\text{-}\beta\textsc{-Alt}\quad \frac{\Gamma \vdash t_1 :_{w_1} T_1 \qquad \Gamma \vdash t_2 :_{w_2} T_2}{\pi_i \langle t_1, t_2 \rangle = t_i :_{w_1 \wedge w_2} : T_i}$$

F. Function types

Now we give the rules for function types, which are also straightforward.

$$\rightarrow\textsc{-Form}\quad \frac{S \text{ type} \qquad T \text{ type}}{S \to T \text{ type}} \qquad\qquad \rightarrow\textsc{-Intro}\quad \frac{\Gamma, x : S \vdash t :_w T}{\Gamma \vdash \lambda x.t :_w S \to T}$$

$$\rightarrow\textsc{-Elim}\quad \frac{\Gamma \vdash f :_w S \to T}{\Gamma, x : S \vdash fx :_w T} \qquad\qquad \rightarrow\text{-}\beta\quad \frac{\Gamma, x : S \vdash t :_w T}{\Gamma, x : S \vdash (\lambda x.t)x = t :_w T}$$

$$\rightarrow\text{-}\eta\quad \frac{\Gamma \vdash f :_w S \to T}{\Gamma \vdash \lambda x.fx = f :_w S \to T}$$

Again, when $\mathbb{W} = \mathbb{1}$, these are the rules for function types in the simply typed lambda calculus. More generally, these rules can be seen as saying that function types are taken weightwise.

Definition 2: For a monoidal poset $\mathbb{W}$, we call the rules C-Emp, C-Ext, Var, Wk, Subst, $\mathbb{W}$-Wk, Wtg-Form, Wtg-Intro, Wtg-Elim, Wtg-β, Wtg-η, $\times$-Form, $\times$-Intro, $\times$-Elim, $\times$-β, $\times$-η, $\rightarrow$-Form, $\rightarrow$-Intro, $\rightarrow$-Elim, $\rightarrow$-β, and $\rightarrow$-η the *$\mathbb{W}$-enriched simply typed lambda calculus*. ◀

G. Monoidal categories of weights

We have so far considered a syntax that starts with a monoidal poset $\mathbb{W}$. We can generalize our calculus to one in which that starts with $\mathcal{W}$.

Notation 1: We use the symbols $\mathcal{W}$ and $\mathbb{W}$ throughout this work to help distinguish when we are talking about the general case ($\mathcal{W}$) or the posetal case ($\mathbb{W}$), but note that every monoidal poset is a monoidal category. ◀

For the analogues of the judgments and rules in Section II-A, we remove the judgment $w \leq v$ wt, and we add the following judgments for any two weights w, v.

$$f : w \Rightarrow v \qquad f = g : w \Rightarrow v$$

If we consider the monoidal category $\mathcal{W}$ to be external to our calculus, then we say that $f : W \Rightarrow v$ means $f \in \hom_{\mathcal{W}}(w, v)$ and $f = g : w \Rightarrow v$ means that $f = g$ (provided $f, g : w \Rightarrow v$).

Just as in Section II-A, we could instead give rules that axiomatize monoidal categories together with generating constants and relations. The rules that define monoidal categories would start with the following rules.

$$\mathcal{W}\text{-Refl}\ \frac{w\ \mathsf{wt}}{id_w : w \Rightarrow w} \qquad \mathcal{W}\text{-Trans}\ \frac{f : u \Rightarrow v \qquad g : v \Rightarrow w}{g \circ f : u \Rightarrow w\ \mathsf{wt}} \qquad \mathcal{W}\text{-Prod}\ \frac{v, w\ \mathsf{wt}}{vw\ \mathsf{wt}}$$

$$\mathcal{W}\text{-Id}\ \frac{}{1\ \mathsf{wt}} \qquad \mathcal{W}\text{-IdL}\ \frac{w\ \mathsf{wt}}{\lambda_w : 1w \cong w} \qquad \mathcal{W}\text{-IdR}\ \frac{w\ \mathsf{wt}}{\rho_w : w1 \cong w}$$

$$\mathcal{W}\text{-Ass}\ \frac{u, v, w\ \mathsf{wt}}{\alpha_{u,v,w} : u(vw) \cong (uv)w} \qquad \mathcal{W}\text{-Func}\ \frac{f : t \Rightarrow u \qquad g : v \Rightarrow w}{fg : tv \Rightarrow uw}$$

In the above, the rules with conclusions of the form $f : A \cong B$ stand for four rules with the same hypotheses as the original: one with conclusion $f : A \to B$, one with conclusion $f^{-1} : B \to A$, one with conclusion $f \circ f^{-1} = id_B : B \to B$, one with conclusion $f^{-1} \circ f = id_A : A \to A$. The above rules additionally need to be accompanied by rules asserting (1) the axioms of a category (i.e., asserting that id provides left and right units for $\circ$ and that $\circ$ is associative), (2) that $\otimes$ is functorial (i.e., respects id and $\circ$), (3) that the isomorphisms λ, ρ, and α are natural, (4) that the triangle identity holds, and (5) that the pentagon identity holds. Giving these rules is routine though tedious (i.e., it consists just of writing the rest of the definition of monoidal category in the above form), and since this is not the focus of this work, we refrain from writing them explicitly.

Independently of how we obtain $\mathcal{W}$, we also need to replace the rule $\mathbb{W}$-Wk with one more suited for $\mathcal{W}$. In $\mathbb{W}$-Wk, we gave the same name to the terms in the hypothesis and conclusion. This is relatively harmless since we assume that the weights form a poset. If the weights form a non-posetal category (or perhaps if we want to be more careful in an implementation), then we should rather give the resulting term a name, as below.

$$\mathcal{W}\text{-Wk}\ \frac{\Gamma \vdash t :_w T \qquad f : v \Rightarrow w}{\Gamma \vdash f(t) :_v T}$$

This needs to be accompanied by rules asserting that this is functorial in f, which we do give explicitly here.

$$\mathcal{W}\text{-Wk-id}\ \frac{\Gamma \vdash t :_w T}{\Gamma \vdash id_w(t) = t :_w T}$$

$$\mathcal{W}\text{-Wk-comp}\ \frac{\Gamma \vdash t :_w T \qquad f : v \Rightarrow w \qquad g : u \Rightarrow v}{\Gamma \vdash g(f(t)) = (f \circ g)(t) :_u T}$$

Definition 3: Given a monoidal category $\mathbb{W}$, we call the rules C-Emp, C-Ext, Var, Wk, Subst, $\mathcal{W}$-Wk, Wtg-Form, Wtg-Intro, Wtg-Elim, Wtg-β, Wtg-η, $\times$-Form, $\times$-Intro, $\times$-Elim, $\times$-β, $\times$-η, $\to$-Form, $\to$-Intro, $\to$-Elim, $\to$-β, and $\to$-η the *$\mathcal{W}$-enriched simply typed lambda calculus*. ◀

III. Semantics

In this section, we describe the semantics of the $\mathcal{W}$-enriched simply typed lambda calculus. We show that it is an internal language of cartesian closed categories enriched in $\mathcal{W}$-relative monoidal categories in the sense that there is a sound (Theorem 1) and complete (Theorem 2) interpretation.

A. Definitions and examples

Definition 4: Consider a monoidal category $(\mathcal{W}, 1, \cdot)$. Say that a monoidal category $(\mathcal{V}, I, \otimes)$ together with a strong monoidal functor $\iota : \mathcal{W} \to \mathcal{V}$ is a *$\mathcal{W}$-relative monoidal category*. ◀

Strong monoidal means that the functor preserves the monoidal structure up to isomorphism. Though we explain the pertinent aspects of the theory of enriched categories, for a complete introduction, see [16].

In examples, $\mathcal{W}$ will often be posetal (in which case we will write it as $\mathbb{W}$), and ι will usually be the inclusion of a subcategory.

Example 3: Consider the category of sets with its cartesian monoidal structure $(\mathcal{S}et, *, \times)$, and let $\mathcal{W}$ be the subcategory consisting just of the singleton $*$. ◀

Example 4: Consider the category of pointed sets with the smash product, and let $\mathcal{W}$ be the subcategory consisting just of the singleton $*$. ◀

Example 5: More generally, for a monoidal poset $\mathbb{W}$, consider the monoidal category of $\mathbb{W}$-fuzzy sets $(\mathcal{S}et(\mathbb{W}), (*, 1), \otimes)$. Observe that $\mathbb{W}$ appears as a submonoidal category of $\mathcal{S}et(\mathbb{W})$, where the inclusion $\mathbb{W} \hookrightarrow \mathcal{S}et(\mathbb{W})$ takes $w \mapsto (*, w)$.

When $\mathbb{W} = \mathbb{1}$, we recover (up to isomorphism) the previous example in $\mathcal{S}et$. ◀

We think of the objects of $\mathcal{W}$ as 'shapes' with which we 'probe' the objects of $\mathcal{V}$: that is, for an object $w \in \mathcal{W}$ and an object $V \in \mathcal{V}$, we will focus on the information carried by $\hom_{\mathcal{V}}(\iota w, V)$. In other words, in the categories $\mathcal{V}$ that we study in this work, one is interested in understanding certain features of its objects where by 'features' we mean functors $F : \mathcal{V} \to \mathcal{S}et$, and these functors are representable by ιw for some w.

Example 6 (continuing Example 3): By probing sets with the singleton $*$, we extract their elements. ◀

Example 7 (continuing Example 4): By probing a pointed set with the point $*$, we always pick out its point. This corresponds to the semantics of `undefined` suggested in the introduction. ◀

Example 8 (continuing Example 5): By probing a $\mathbb{W}$-fuzzy set with an object of the form $(*, w)$ – that is, by taking the hom-set $\hom_{\mathcal{S}et(\mathbb{W})}((*, w), (S, f))$ – we obtain the set of all elements of $x \in S$ such that $w \leq f(x)$ – that is, the set of all elements with weight at least w. ◀

Example 9: Consider a closed monoidal category $(\mathcal{V}, \otimes, I)$ with an interval object: that is, an object J together with two morphisms $0, 1 : I \to J$. Examples include the category of categories with the walking arrow, any convenient category of topological spaces with the unit interval $[0, 1]$, the category of simplicial sets with Δ^1, and any of the categories of cubical sets with $\square^1$. Let $\mathcal{J}$ denote the submonoidal category generated by $0, 1 : I \to J$; denote its objects as J^n (where $J^0 := I$). Then one can understand an object V by calculating $\hom_{\mathcal{V}}(J^0, V)$ (the set of 'points' of V), $\hom_{\mathcal{V}}(J, V)$ (the set of 'intervals' of V),$\hom_{\mathcal{V}}(J^2, V)$ (the set of 'squares' of V), etc. This is a very useful set of tools when working in such categories, and in this work we formalize it. ◀

Example 10: Consider a small, monoidal category $(\mathcal{W}, 1, \cdot)$. Day convolution [8] produces a closed monoidal structure on the category $\widehat{\mathcal{W}}$ of presheaves on $\mathcal{W}$ such that the Yoneda embedding $よ : \mathcal{W} \hookrightarrow \widehat{\mathcal{W}}$ is strong monoidal.

The main approach to working with presheaves probes them with representables: i.e., one studies a presheaf P by calculating $\hom_{\mathcal{V}}(よ(w), P)$ for each $w \in \mathcal{W}$. ◀

Example 11: The category $\mathcal{CPO}$ of complete partial orders and continuous functions is cartesian closed as is the category $\mathcal{DCPO}$ of directed complete partial orders [1]. In either of these categories, one can extract the underling set of an object X by considering $\hom(F*, X)$ where $F*$ denotes the free object on one generator, or ω-chains by considering $\hom(\omega, X)$ where ω denotes the free omega chain, etc. ◀

$\mathcal{W}$-relative monoidal categories appear in this work as the base of enrichment of the categories which will interpret the $\mathcal{W}$-enriched simply typed lambda calculus. If $\mathcal{V}$ is monoidal closed, then it can be seen as a category enriched in itself [16, § 1.6], and these will serve as our most basic examples of enriched categories. As stated, the examples given above are cartesian or monoidal closed, with the exception of $\mathcal{S}et(\mathbb{W})$ (in generality).

Example 12 (continuing Example 4): The smash monoidal structure on pointed sets is closed. ◀

Example 13 (continuing Example 5): If the monoidal poset $\mathbb{W}$ is complete, then so is $\mathcal{S}et(\mathbb{W})$: the product $\prod_i (S_i, f_i)$ is given by $(\prod_i S_i, \lambda x. \wedge_i f_i(x))$.

If the monoidal poset $\mathbb{W}$ is also closed, then so is $\mathcal{S}et(\mathbb{W})$. The exponential $(S, f)^{(T,g)}$ is given by $(S^T, \lambda\alpha. \wedge_{t\in T} f\alpha(t)^{g(t)})$. ◀

Interesting examples of $\mathcal{V}$-enriched categories other than $\mathcal{V}$ itself include the following.

Example 14: Suppose that $\mathcal{V}$ is monoidal closed and complete. Given a set S, the collection of functions $S \to \mathrm{ob}(\mathcal{V})$ naturally inherits a $\mathcal{V}$-category structure, where $\hom_{S\to\mathrm{ob}(\mathcal{V})}(f, g) := \prod_{x\in S} \underline{\hom}_{\mathcal{V}}(fx, gx)$ (where $\underline{\hom}_{\mathcal{V}}$ denotes the internal hom of $\mathcal{V}$ given by its closed structure).

This is (with some assumptions) a consequence of the more general fact that for any $\mathcal{V}$-enriched category $\mathcal{C}$, there is a presheaf $\mathcal{V}$-category $\widehat{\mathcal{C}} := [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ [16, § 2.2]. ◀

Example 15: Simplicially enriched categories (categories enriched in simplicial sets) are extremely important in modern homotopy theory, as a subcategory of these constitutes one model of the theory of $(\infty, 1)$-categories. When studying these, it is common to calculate the points, the intervals, etc. in each hom-space, as they correspond to the 1-cells, the 2-cells, etc. in the corresponding $(\infty, 1)$-categories. ◀

Example 16: Similarly, categories enriched in the category of categories are 2-categories (or to be general, categories enriched in n-categories are $n+1$-categories). There, the points are the 1-cells, the intervals are the 2-cells, etc. ◀

Example 17: We obtain another interesting perspective on directed type theory in the following way. They (e.g., [23, 22, 14]) often come with some number of modes, usually at least $-, +$, which correspond to contravariant functors and covariant functors, respectively, of (higher) categories. There is a group structure on $\mathbb{Z}_2 := \{+, -\}$ where $+$ is the identity and $-^2 = +$. Considered as a discrete poset, this is a monoidal one.

Then consider the following $\mathcal{S}et(\mathbb{Z}_2)$-category of categories. Let the objects be categories, and let $\hom(\mathcal{C}, \mathcal{D})$ be the collection of all contravariant functors *and* covariant functors from $\mathcal{C} \to \mathcal{D}$. There is a function $\hom(\mathcal{C}, \mathcal{D}) \to \mathbb{Z}_2$ that indicates the variance of each functor. This is indeed a $\mathcal{S}et(\mathbb{Z}_2)$-category: the (covariant) identity functor $\mathcal{C} \to \mathcal{C}$ gives us a morphism $(*, +) \to \hom(\mathcal{C}, \mathcal{C})$, and for two functors $F \in \hom(\mathcal{C}, \mathcal{D})$ and $G \in \hom(\mathcal{C}, \mathcal{D})$ with variances v, w respectively, the composite $G \circ F$ is a functor with variance vw, giving composition in this $\mathcal{S}et(\mathbb{Z}_2)$-category.

While $-$ corresponds to the operation $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{op}}$, and $+$ corresponds to $\mathcal{C} \mapsto \mathcal{C}$, other modalities, such as those corresponding to $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{core}}$ and the localization of a category (by all of its morphisms) are often also considered. With these, one gets a larger, non-discrete monoidal poset (that is cartesian closed), and this approach extends to that setting.

We expect this perspective to also extend to the settings of other modal logics, and we leave it to future work to concretely compare the perspective developed here based on enriched categories, and others: for instance the ones based on a 2-category of modes explored in [19, 13]. ◀

Example 18 (continuing Example 11): $\mathcal{CPO}$-categories were used by Fiore to develop axiomatic domain theory [9]. $\mathcal{CPO}$-categories and $\mathcal{DCPO}$-categories have also appeared recently in the literature to develop reversible functional programming languages (see e.g. [15, 6]). ◀

B. Soundness

In the next subsections, we prove the following theorem.

Theorem 1 (Enriched Lambek interpretation): Consider a monoidal category $\mathcal{W}$ and a $\mathcal{W}$-relative monoidal category $\mathcal{V}$ with finite products and which is powered by $\mathcal{W}$. There is an interpretation of the $\mathcal{W}$-enriched simply typed lambda calculus in any cartesian closed $\mathcal{V}$-category.

If $\mathcal{W}$ is a monoidal poset $\mathbb{W}$, this restricts to an interpretation of the $\mathbb{W}$-enriched simply typed lambda calculus in any cartesian closed $\mathcal{V}$-category. ◀

Proof: See Lemmas 1,2,3,4. ■

We define the adjective 'cartesian closed' as the following. The concept of powered and the constituents of cartesian closed – finite products, unary products weighted by $\mathcal{W}$, and exponentials – will be defined in the corresponding subsections that follow.

Definition 5: We call a $\mathcal{V}$-category *cartesian closed* if it has all finite products, unary products weighted by $\mathcal{W}$, and exponentials. ◀

Example 19: Recall that all examples for $\mathcal{V}$ that we have considered are monoidal closed (which implies powered by $\mathbb{W}$) and have all finite products (except $\mathcal{S}et(\mathbb{W})$ in general – for these we need to assume that $\mathbb{W}$ is closed and has all finite meets). Thus, these meet our hypotheses on the $\mathcal{W}$-relative monoidal category of Theorem 1.

We will not attempt to enumerate the cartesian closed $\mathcal{V}$-categories, but just as $\mathcal{S}et$ is the primordial cartesian closed $\mathcal{S}et$-category, each $\mathcal{V}$ itself that is monoidal closed (ensuring unary weighted products and exponentials) and has all finite products also meets our hypotheses on the $\mathcal{V}$-category of Theorem 1. ◀

Example 20 (continuing Example 3): When $\mathcal{V}$ is $\mathcal{S}et$ and $\mathcal{W}$ is the subcategory consisting just of the singleton, we recover from Theorem 1 the usual interpretation of the simply typed lambda calculus in cartesian closed categories. ◀

Remark 1: There are two categorical approaches to interpreting the simply typed lambda calculus. In the more common one, contexts $T_1, ..., T_n$ are interpreted as products $\prod_{i=1}^{n}[T_i]$, so the interpreting category needs to have all finite products. This is not much of an assumption, since the binary product type formers require the category to all non-trivial finite products anyway.

There is another approach, more faithful to the syntax (well explicated in [26]). In that approach, the syntax is interpreted in *multi*categories and this extra structure is used to interpret contexts.

We view the latter as a better approach that gives a clearer comparison between the syntax and the semantics, but we follow the former in this short work. ◀

Thus, for the rest of this section, fix a $\mathcal{W}$-relative monoidal category $\mathcal{V}$ with finite products and a $\mathcal{V}$-category $\mathcal{C}$ with finite products. Note that all $\mathcal{V}$ we gave above in examples have finite products (with the exception of $\mathcal{S}et(\mathbb{W})$ – there we need $\mathbb{W}$ to have finite products).

C. Interpretation of structural rules

We interpret Γ ctx and T type as objects $[\Gamma]$ and $[T]$ in $\mathcal{C}$. As with the usual simply typed lambda calculus, $[\diamond]$ is interpreted as the terminal object and for objects $[\Gamma]$ and $[T]$, we take $[\Gamma, x : T] := [\Gamma] \times [T]$.

Notation 2: Because our interpretation of terms will use hom-sets of the form $\hom_{\mathcal{V}}(\iota w, \hom_{\mathcal{C}}(X, Y))$, we introduce a notation for this (on the semantic side). Let $X \to_w Y$ denote the set $\hom_{\mathcal{V}}(\iota w, \hom_{\mathcal{C}}(X, Y))$. We write $f : X \to_w Y$ for $f \in \hom_{\mathcal{V}}(\iota w, \hom_{\mathcal{C}}(X, Y))$.

The identity on an object X in $\mathcal{C}$ is given by a morphism $\mathrm{id}_X \in \hom_{\mathcal{V}}(I, \hom_{\mathcal{C}}(X, Y))$ in $\mathcal{V}$. We can now write this as $\mathrm{id}_X : X \to_1 X$.

Similarly, composition is given as a morphism $\circ_{X,Y,Z}$: $\hom_{\mathcal{C}}(X, Y) \otimes \hom_{\mathcal{C}}(Y, Z) \to \hom_{\mathcal{C}}(X, Y)$ in $\mathcal{V}$. Morphisms $f : X \to_v Y$ and $g : Y \to_w Z$ can then be combined to give a morphism $\iota(vw) \to \hom_{\mathcal{C}}(X, Y) \otimes \hom_{\mathcal{C}}(Y, Z)$ in $\mathcal{V}$; composing with $\circ_{X,Y,Z}$ then produces a morphism that we will write as $g \circ f : X \to_{vw} Z$. ◀

Example 21 (continuing Example 17): We now obtain a pleasing notation with a concrete mathematical meaning for covariant and contravariant functors. We write $\mathcal{C} \to_{-} \mathcal{D}$ for contravariant functors and $\mathcal{C} \to_{+} \mathcal{D}$ for covariant functors. ◀

From here on, we will stop writing the interpretation function $[\ ]$ for readability.

We interpret a judgment $\Gamma \vdash t :_w T$ as $t : \Gamma \to_w T$.

Given a product $\Gamma \times T \times \Delta$, we interpret VAR as the morphism $\Gamma \times T \times \Delta \to_1 T$ given by the following derivation. For an arbitrary product $A \times B$ in $\mathcal{C}$, observe that we have a projection $\pi_1 : A \times B \to_1 A$ given as the composition

$$\begin{aligned} I &\xrightarrow{id_{A\times B}} \hom_{\mathcal{C}}(A \times B, A \times B) \\ &\cong \hom_{\mathcal{C}}(A \times B, A) \times \hom_{\mathcal{C}}(A \times B, B) \\ &\to \hom_{\mathcal{C}}(A \times B, A) \end{aligned}$$

where the first morphism is the identity on $A \times B$, the isomorphism is an instantiation of the universal property of $A \times B$, and the last morphism is the product projection in $\mathcal{V}$. Similarly, we have a projection $\pi_2 : A \times B \to_1 B$ and a projection $\pi_T : \Gamma \times T \times \Delta \to_1 T$.

Given an element $t : \Gamma \times \Delta \to_w T$ and a type S, we interpret WK as the morphism $t : \Gamma \times S \times \Delta \to_w T$ constructed in the following way. As above, we have a product projection $\pi_{\Gamma\times\Delta} : \Gamma \times S \times \Delta \to_1 \Gamma \times \Delta$. We postcompose this with t to get a morphism $\Gamma \times S \times \Delta \to_w T$.

Given morphisms $t : \Gamma \times S \times \Delta \to_w T$ and $s : \Gamma \to_v S$, we interpret SUBST as the morphism $t[s/x] : \Gamma \times \Delta \to_{vw} T$ given by the following construction. First, analogous to the product projections described above, there is a morphism $\delta_\Gamma \times \mathrm{id}_\Delta$: $\Gamma \times \Delta \to_1 \Gamma \times \Gamma \times \Delta$. Similarly, we can construct from s a morphism $\mathrm{id}_\Gamma \times s \times \mathrm{id}_\Delta : \Gamma \times \Gamma \times \Delta \to_v \Gamma \times S \times \Delta$. Now we compose $\delta_\Gamma \times \mathrm{id}_\Delta$ with $\mathrm{id}_\Gamma \times s \times \mathrm{id}_\Delta$ and t to obtain a morphism $t[s/x] : \Gamma \times \Delta \to_{vw} T$.

To interpret $\mathcal{W}$-WK, we start with a morphism $t : \Gamma \to_w T$ and a morphism $f : v \to w$ in $\mathcal{W}$. We precompose the morphism $t : w \to \hom_{\mathcal{C}}(\Gamma, T)$ with f to obtain a new morphism $f(t) : v \to \hom_{\mathcal{C}}(\Gamma, T)$. Since $t \circ id_w = t$ and $t \circ (f \circ g) = (t \circ f) \circ g$, we can also interpret $\mathcal{W}$-WK-ID and $\mathcal{W}$-WK-COMP.

In the case that $\mathcal{W}$ is a poset $\mathbb{W}$, then we abuse notation, also calling $f(t)$ by t, and we thus interpret $\mathbb{W}$-WK.

Now we have shown the following.

Lemma 1: There is an interpretation of C-EMP, C-EXT, VAR, WK, SUBST, $\mathcal{W}$-WK, $\mathcal{W}$-WK-ID, and $\mathcal{W}$-WK-COMP into $\mathcal{C}$.

If $\mathcal{W}$ is a monoidal poset $\mathbb{W}$, then this restricts to an interpretation of C-EMP, C-EXT, VAR, WK, SUBST, and $\mathbb{W}$-WK in $\mathcal{C}$. ◀

D. Interpretation of rules for weightings of types

To express the universal property for unary products weighted by elements of $\mathcal{W}$, the intended interpretation of the weightings of types, we first need to suppose that $\mathcal{V}$ is *powered* by $\mathcal{W}$: that is, that for every object $V \in \mathcal{V}$ and every $w \in \mathcal{W}$, there is an object V^w with the following universal property.

$$\hom_{\mathcal{V}}(Z \otimes \iota w, V) \cong \hom_{\mathcal{V}}(Z, V^w)$$

Example 22: Note that if $\mathcal{V}$ is closed, then it is powered by $\mathcal{W}$. In all examples of $\mathcal{V}$ given above, $\mathcal{V}$ is closed (with the exception of $\mathcal{S}et(\mathbb{W})$ – there we need $\mathbb{W}$ to be closed). ◀

Now we define unary products weighted by elements of $\mathcal{W}$. Given an object $T \in \mathcal{C}$ and an object $w \in \mathcal{W}$, these are objects with the following universal property.

$$\hom_{\mathcal{C}}(\Gamma, T^w) \cong \hom_{\mathcal{C}}(\Gamma, T)^w$$

Assuming such a T^w, we calculate for any $v \in \mathbb{W}$:

$$\begin{aligned} \hom_{\mathcal{V}}(\iota v, \hom_{\mathcal{C}}(\Gamma, T^w)) &\cong \hom_{\mathcal{V}}(\iota v, \hom_{\mathcal{C}}(\Gamma, T)^w) \\ &\cong \hom_{\mathcal{V}}(\iota(vw), \hom_{\mathcal{C}}(\Gamma, T)) \end{aligned}$$

where the first bijection uses the universal property of T^w and the second is the universal property of $\hom_{\mathcal{C}}(\Gamma, T)^w$. Thus, morphisms $\Gamma \to_v T^w$ are in bijection with morphisms $\Gamma \to_{vw} T$.

Now the rules can be easily interpreted: WTG-FORM corresponds to the existence of the object T^w, WTG-INTRO and WTG-ELIM then express the two functions between $\Gamma \to_v T^w$ and $\Gamma \to_{vw} T$, and WTG-β and WTG-η express that they are inverses of each other.

Lemma 2: Suppose that $\mathcal{V}$ is powered by $\mathcal{W}$, and suppose that $\mathcal{C}$ has unary products weighted by $\mathcal{W}$. There is an interpretation of WTG-FORM, WTG-INTRO, WTG-ELIM, WTG-β, WTG-η into $\mathcal{C}$. ◀

Example 23: When $\mathcal{W}$ is $\mathbb{1}$, we can only weight a type T by 1, and this weighting is isomorphic to T. Thus, when $\mathcal{V}$ is $\mathcal{S}et$, these type formers do not add anything to the categorical structure described by the simply typed lambda calculus. Compare this with the analogous observation on the syntactic side in Example 2. ◀

E. Interpretation of rules for binary products

The universal property for binary products is as usual.

$$\hom_{\mathcal{C}}(\Gamma, T_1 \times T_2) \cong \hom_{\mathcal{C}}(\Gamma, T_1) \times \hom_{\mathcal{C}}(\Gamma, T_2)$$

For any $w \in \mathbb{W}$, we find

$$\begin{aligned} &\hom_{\mathcal{V}}(\iota w, \hom_{\mathcal{C}}(\Gamma, T_1 \times T_2)) \\ &\cong \hom_{\mathcal{V}}(\iota w, \hom_{\mathcal{C}}(\Gamma, T_1) \times \hom_{\mathcal{C}}(\Gamma, T_2)) \\ &\cong \hom_{\mathcal{V}}(\iota w, \hom_{\mathcal{C}}(\Gamma, T_1)) \times \hom_{\mathcal{V}}(\iota w, \hom_{\mathcal{C}}(\Gamma, T_2)) \end{aligned}$$

where the first bijection uses the universal property of binary products in $\mathcal{C}$, and the second is the universal property of binary products in $\mathcal{V}$. Now we see that morphisms $\Gamma \to_w T_1 \times T_2$ are in bijection with pairs of morphisms $\Gamma \to_w T_1$ and $\Gamma \to_w T_2$. This is exactly what is expressed by the rules for binary products, so we have the following result.

Lemma 3: There is an interpretation of the rules $\times$-FORM, $\times$-INTRO, $\times$-ELIM, $\times$-β, and $\times$-η into $\mathcal{C}$. ◀

F. Interpretation of rules for function types

We use the usual universal property for exponentials:

$$\hom_{\mathcal{C}}(\Gamma \times S, T) \cong \hom_{\mathcal{C}}(\Gamma, T^S).$$

Now we simply observe that

$$\hom_{\mathcal{V}}(\iota w, \hom_{\mathcal{C}}(\Gamma \times S, T)) \cong \hom_{\mathcal{V}}(\iota w, \hom_{\mathcal{C}}(\Gamma, T^S))$$

so morphisms $\Gamma \times S \to_w T$ correspond bijectively to $\Gamma \to_w T^S$.

Thus we interpret $[S \to T] := T^S$, and we see that the rules for function types are exactly describing the above bijection.

Lemma 4: Suppose that $\mathcal{C}$ has exponentials. There is an interpretation of the rules $\to$-FORM, $\to$-INTRO, $\to$-ELIM, $\to$-β, $\to$-η into $\mathcal{C}$. ◀

This concludes the interpretation of the syntax into $\mathcal{C}$, summarized in Theorem 1 above.

G. The syntactic category and completeness

For the rest of this section, we assume the $\mathcal{W}$-category $\mathcal{V}$ is finitely complete and powered by $\mathcal{W}$.

The syntax of the $\mathcal{W}$-enriched simply typed lambda calculus (perhaps augmented with some constants) naturally forms a $\widehat{\mathcal{W}}$-category (see Example 10).

Construction 1: Let C be a collection of constants to be added to the $\mathcal{W}$-enriched simply typed lambda calculus. We define the syntactic $\widehat{\mathcal{W}}$-category $\mathcal{S}_C$ as follows.

Let the collection $\mathrm{ob}(\mathcal{S}_C)$ of objects be the collection of types.

For an object $w \in \mathcal{W}$ and types T, T', let $\hom_{\mathcal{S}_C}(T, T')(w)$ be the set of terms of the form $T \vdash t :_w T'$. For $f : v \to w$ in $\mathcal{W}$, $\mathcal{W}$-WK gives the restriction map $f(-) : \hom_{\mathcal{S}_C}(T, T')(w) \to \hom_{\mathcal{S}_C}(T, T')(v)$. The rules $\mathcal{W}$-WK-ID, and $\mathcal{W}$-WK-COMP ensure that this is a functor $\hom_{\mathcal{S}_C}(T, T') : \mathcal{W}^{\mathrm{op}} \to \mathcal{S}et$.

To show this is a $\widehat{\mathcal{W}}$-category, we first need identities: that is, maps $よ(1) \to \hom_{\mathcal{S}_C}(T, T)$ for each type T. The rule VAR gives an element in $\hom_{\mathcal{S}_C}(T, T)(1)$, which is in bijection with the set of morphisms $よ(1) \to \hom_{\mathcal{S}_C}(T, T)$ by the Yoneda Lemma.

We furthermore need to establish composition: that is, maps $\hom_{\mathcal{S}_C}(S, T) \otimes \hom_{\mathcal{S}_C}(T, U) \to \hom_{\mathcal{S}_C}(S, U)$. It suffices to define functions $(\hom_{\mathcal{S}_C}(S, T) \otimes \hom_{\mathcal{S}_C}(T, U))(w) \to \hom_{\mathcal{S}_C}(S, U)(w)$ natural in w. By definition,

$$\begin{aligned} &(\hom_{\mathcal{S}_C}(S, T) \otimes \hom_{\mathcal{S}_C}(T, U))(w) \\ &:= \mathrm{colim}_{\substack{v_1, v_2 \in \mathbb{W} \\ w \leq v_1 v_2}} \hom_{\mathcal{S}_C}(S, T)(v_1) \times \hom_{\mathcal{S}_C}(S, T)(v_2). \end{aligned}$$

Thus, it suffices to define natural functions $\hom_{\mathcal{S}_C}(S, T)(v_1) \times \hom_{\mathcal{S}_C}(S, T)(v_2) \to \hom_{\mathcal{S}_C}(S, U)(w)$ for $w \leq v_1 v_2$. But SUBST gives us a function $\hom_{\mathcal{S}_C}(S, T)(v_1) \times \hom_{\mathcal{S}_C}(S, T)(v_2) \to \hom_{\mathcal{S}_C}(S, U)(v_1 v_2)$

and $\mathcal{W}$-WK gives us a function $\hom_{\mathcal{S}_C}(S,U)(v_1v_2) \to \hom_{\mathcal{S}_C}(S,U)(w)$, so we compose these. Naturality is again given by $\mathcal{W}$-WK together with $\mathcal{W}$-WK-ID and $\mathcal{W}$-WK-COMP and one of the usual unwritten rules governing implicit substitution: that $f(t)[s/x] = f(t[s/x])$ since x as a term variable cannot appear in the morphism f of $\mathcal{W}$.

The rules governing weighting of types, binary products, and function types say exactly that $\mathcal{S}_C$ is cartesian closed. ◀

From the existence of the syntactic category, we automatically get completeness.

Lemma 5: The interpretation of Theorem 1 is complete for $\widehat{\mathcal{W}}$-categories: if a judgment holds in all of the interpretations given there in cartesian closed $\widehat{\mathcal{W}}$-categories $\mathcal{C}$, then it holds in the $\mathcal{W}$-enriched simply typed lambda calculus. ◀

Proof: Since such a judgment in particular holds in the syntactic category, it also holds in the syntax. ■

We can weaken the hypothesis of this statement to get the following.

Theorem 2: The interpretation of Theorem 1 is complete: if a judgment holds in all of the interpretations given there in cartesian-closed $\mathcal{V}$-categories $\mathcal{C}$ for all $\mathcal{W}$-relative monoidal categories $\mathcal{V}$, then it holds in the $\mathcal{W}$-enriched simply typed lambda calculus. ◀

We have not defined a category of all $\mathcal{V}$-enriched categories for all $\mathcal{W}$-relative monoidal categories $\mathcal{V}$, so we will not say that $\mathcal{S}_C$ is initial among all such categories. However, $\mathcal{S}_C$ is initial among all cartesian closed $\widehat{\mathcal{W}}$-categories with the constants C.

Corollary 1: There is a unique (up to isomorphism) structure-preserving (up to isomorphism) $\widehat{\mathcal{W}}$-functor from $\mathcal{S}_C$ to any cartesian closed $\widehat{\mathcal{W}}$-category with the constants C.

Given an interpretation of the syntax with constants C in a $\widehat{\mathcal{W}}$-category $\mathcal{C}$, there is a unique (up to equality) $\widehat{\mathcal{W}}$-functor $\mathcal{S}_C \to \mathcal{C}$ that preserves the interpretation (up to equality). ◀ By structure-preserving, we mean that it preserves the constants C and the cartesian closed structure up to isomorphism.

Proof: The proof of Theorem 1 gives the action of the functor on objects. It also gives the action of the functor on the components $\hom_{\mathcal{S}_C}(A,B)(w)$ of the hom-presheaves. By $\mathcal{W}$-WK, the action on these components assembles into a morphism of presheaves. This functor preserves identities and composition by construction. It is also structure-preserving by construction.

By interpretation, we mean an assignment of judgments to particular (i.e., not isomorphism classes of) components of $\mathcal{C}$. Thus, it constitutes a choice of the cartesian closed structure that is otherwise only given up to isomorphism and this choices can be used to specify the above functor not just up to isomorphism, but up to equality. ■

H. Completeness with respect to $\mathcal{S}et(\mathbb{W})$-categories

Even though our initiality result is confined only to $\widehat{\mathcal{W}}$-categories, we also might wonder about completeness with respect to $\mathcal{V}$-categories for other $\mathcal{V}$. We do not give a complete analysis here, but we discuss how this could be approached by working out the example $\mathcal{V} := \mathcal{S}et(\mathbb{W})$.

Remark 2: By focusing on the sets $\hom_{\mathcal{V}}(\iota w, V)$, we are focusing on the restricted Yoneda embedding $\text{よ}_\iota : \mathcal{V} \to \widehat{\mathcal{W}}$ that sends $V \mapsto \hom_{\mathcal{V}}(\iota -, V)$. That is, we can understand the interpretation of our syntax in a $\mathcal{V}$-category $\mathcal{C}$ given in Theorem 1 as comprising two steps: construct a $\widehat{\mathcal{W}}$-category $\text{よ}_\iota^* \mathcal{C}$ by changing the base of enrichment via よ_ι, and interpret the syntax there. In this light, the focus on $\widehat{\mathcal{W}}$ in the previous subsection is not surprising. ◀

Now, we show that $\mathcal{S}et(\mathbb{W})$ can be regarded a full subcategory of $\widehat{\mathbb{W}}$ and characterize its image.

Proposition 1: The objects in the image of $\text{よ}_\iota : \mathcal{S}et(\mathbb{W}) \to \widehat{\mathbb{W}}$ are those presheaves P with the following properties:

1) the restriction maps of P are monic (so we can think of them as subset inclusions); and
2) for all $w \in \mathbb{W}$ and $x \in P(w)$, the subset $\{v \mid x \in P(v)\} \subseteq \mathbb{W}$ has a maximal element.

Note that if $\mathbb{W}$ is cocomplete, the second requirement is equivalent to the requirement that P preserves non-empty limits. ◀

Proof: Since $\text{よ}_\iota(S,\alpha)(w)$ is the subset of S consisting of those elements x such that $w \leq \alpha(x)$, we write it as $S_{\geq w}$ for clarity in these proofs.

To see that the objects in the image have these properties, note that the restriction map corresponding to a $v \leq w$ is an inclusion $S_{\geq w} \subseteq S_{\geq v}$. Given a $w \in \mathbb{W}$ and $x \in S_{\geq w}$, the subset $\{v \mid x \in P(v)\}$ has a maximal element: $\alpha(x)$.

Now, consider a presheaf P satisfying the two properties above. Let S be the colimit of P, and let $\alpha : S \to \mathbb{W}$ take $x \in S$ to the maximum of $\{v \mid x \in P(v)\}$. We have $S_{\geq -} = P$, so よ_ι is surjective on presheaves with these two properties. ■

Proposition 2: The restricted Yoneda embedding よ_ι : $\mathcal{S}et(\mathbb{W}) \to \widehat{\mathbb{W}}$ is fully faithful. In other words, $\iota : \mathbb{W} \to \mathcal{S}et(\mathbb{W})$ is dense. ◀

Proof: Here, for $f : (S,\alpha) \to (T,\beta)$, we also write $f_{\geq w} := \text{よ}_\iota f(w) : S_{\geq w} \to T_{\geq w}$.

For faithfulness, consider $f, g : (S,\alpha) \to (T,\beta)$ such that $f_{\geq w} = g_{\geq w}$ for all $w \in \mathbb{W}$ and $x \in f$. Then $x \in S_{\geq \alpha(x)}$, and $f_{\geq \alpha(x)}(x) = g_{\geq \alpha(x)}(x)$. But $f_{\geq \alpha(x)}, g_{\geq \alpha(x)}$ are restrictions of f, g to $S_{\geq w}$, so $f(x) = g(x)$, and we conclude $f = g$.

For fullness, consider a $g(-) : S_{\geq -} \to T_{\geq -}$. Given $x \in X$, define $f : S \to T$ to take x to $g(\alpha x)x$. This is a morphism of $\mathbb{W}$-fuzzy sets, since $g(\alpha x)x \in T_{\geq \alpha x}$. Now for any $w \in \mathbb{W}$ and $x \in S_{\geq w}$, we have $f_{\geq w}(x) := f(x) := g(\alpha x)(x)$ (using Proposition 1 to think of these elements as all living in subsets of T). But by naturality, $g(\alpha x)(x) = g(w)(x)$. Thus $f_{\geq w}(x) = g(w)(x)$, so $f_{\geq -} = g$. ■

Since we have identified $\mathcal{S}et(\mathbb{W})$ with a full subcategory of $\widehat{\mathbb{W}}$, we can add rules to the syntax corresponding to the conditions in Proposition 1 to restrict the interpretation to this subcategory, making the resulting syntax complete with respect to it. That is, we could add the following rules.

$$\frac{\Gamma \vdash s, t :_w T \qquad v \leq w \text{ wt} \qquad \Gamma \vdash s = t :_v T}{\Gamma \vdash s = t :_w T}$$

$$\frac{\Gamma \vdash t :_w T}{\max(t) \text{ wt}} \qquad \frac{\Gamma \vdash t :_w T}{w \leq \max(t) \text{ wt}} \qquad \frac{\Gamma \vdash t :_w T}{\Gamma \vdash t :_{\max(t)} T}$$

Theorem 3: There is an interpretation of the $\mathbb{W}$-enriched simply typed lambda calculus together with the above rules in cartesian closed $\mathcal{S}et(\mathbb{W})$-categories that extends the interpretation given by Theorem 1.

Moreover, this interpretation is complete. ◀

IV. The Curry-Howard correspondence and intuitionistic fuzzy logic

Here, we extract from the $\mathbb{W}$-enriched simply typed lambda calculus a corresponding (in the sense of Curry-Howard) $\mathbb{W}$-enriched natural deduction. We argue that we obtain a sound and complete natural deduction system for fuzzy logic, and we compare this with other proposals in the literature.

A. Syntax of enriched natural deduction

We start with a monoidal poset $\mathbb{W}$ as before. We could start with a monoidal category $\mathcal{W}$, but the posetal case is more congruent with the setting of natural deduction.

We have the following judgments.

$$w \text{ wt} \qquad v \leq w \text{ wt} \qquad \Gamma \text{ ctx} \qquad T \text{ type} \qquad \Gamma \vdash_w T$$

If we want to reason about equality (e.g., to compare T^{1w} and T^w), then we will also need corresponding equality judgments; however, we do not need them for the presentation of the rules.

Now we erase terms from the rules of the $\mathbb{W}$-enriched simply typed lambda calculus and use more logical symbols to obtain the following rules.

C-Emp
$$\frac{}{\Diamond \text{ ctx}}$$

C-Ext
$$\frac{\Gamma \text{ ctx} \qquad T \text{ type}}{\Gamma, T \text{ ctx}}$$

Var
$$\frac{\Gamma, T, \Delta \text{ ctx}}{\Gamma, T, \Delta \vdash_1 T}$$

Wk
$$\frac{\Gamma, \Delta \vdash_w T \qquad S \text{ type}}{\Gamma, S, \Delta \vdash_w T}$$

Subst
$$\frac{\Gamma, S, \Delta \vdash_w T \qquad \Gamma \vdash_v S}{\Gamma, \Delta \vdash_{vw} T}$$

W-Wk
$$\frac{\Gamma \vdash_w T \qquad v \leq w \text{ wt}}{\Gamma \vdash_v T}$$

Wtg-Form
$$\frac{T \text{ type} \qquad w \text{ wt}}{T^w \text{ type}}$$

Wtg-Intro
$$\frac{\Gamma \vdash_{vw} T \qquad w \text{ wt}}{\Gamma \vdash_v T^w}$$

Wtg-Elim
$$\frac{\Gamma \vdash_v T^w}{\Gamma \vdash_{vw} T}$$

∧-Form
$$\frac{T_1 \text{ type} \qquad T_2 \text{ type}}{T_1 \wedge T_2 \text{ type}}$$

∧-Intro
$$\frac{\Gamma \vdash_w T_1 \qquad \Gamma \vdash_w T_2}{\Gamma \vdash_w T_1 \wedge T_2}$$

∧-Elim
$$\frac{\Gamma \vdash_w T_1 \wedge T_2}{\Gamma \vdash_w T_i}$$

⇒-Form
$$\frac{S \text{ type} \qquad T \text{ type}}{S \Rightarrow T \text{ type}}$$

⇒-Intro
$$\frac{\Gamma, S \vdash_w T}{\Gamma \vdash_w S \Rightarrow T}$$

⇒-Elim
$$\frac{\Gamma \vdash_w S \Rightarrow T}{\Gamma, S \vdash_w T}$$

Definition 6: Call the collection of the rules above *$\mathbb{W}$-enriched natural deduction.* ◀

Example 24: Note that when $\mathbb{W} = \mathbb{1}$ these are equivalent to the rules for natural deduction. ◀

Now, as with the simply typed lambda calculus, it is evident by construction that there is a derivation of a formula $\Gamma \vdash_w T$ in natural deduction if and only if there is a construction of a term $\Gamma \vdash t :_w T$ (modulo the different logical symbols). If we want a true isomorphism between the sets of derivations and the sets of terms, then as usual, we also need to add rules equating derivations that correspond to $\mathcal{W}$-Wk-Id, $\mathcal{W}$-Wk-Comp, and the β and η rules of the $\mathbb{W}$-enriched simply typed lambda calculus.

Theorem 4 (Enriched Curry-Howard correspondence): There is a correspondence between the derivations of formulas in $\mathbb{W}$-enriched natural deduction and the terms in the $\mathbb{W}$-enriched simply typed lambda calculus.

Assuming equations of derivations corresponding to $\mathcal{W}$-Wk-Id, $\mathcal{W}$-Wk-Comp, and the β and η rules of the $\mathbb{W}$-enriched simply typed lambda calculus, this correspondence becomes an isomorphism. ◀

B. Semantics of enriched natural deduction

The Curry-Howard correspondence is usually presented as rather self-evident on the syntactical side, as we have done above. However, there is also a Curry-Howard correspondence on the semantical side. With the perspective of enriched category theory, we understand the *usual* Curry-Howard correspondence between the simply typed lambda calculus and natural deduction in the following semantical way. One can construct from the rules for natural deduction (perhaps augmented with some constants C) a preordered set: take the set of contexts (equivalently, types) as the underlying set and say $\Gamma \leq T$ whenever $\Gamma \vdash T$ is derivable. Then the rules will ensure that this preordered set has binary meets and implications. Alternatively, one can read off from the rules (together with equalities corresponding to β and η rules governing derivations) a category: take the collection of contexts (equivalently, types) as the collection of objects, and let $\hom(\Gamma, T)$ be the set of derivations of $\Gamma \vdash T$. Thus, one obtains a syntactic preordered set $\mathbb{S}_C$ and a syntactic category $\mathcal{S}_C^{\text{nd}}$. We understand the Curry-Howard correspondence to refer to two facts then: (1) that $\mathcal{S}_C^{\text{nd}}$ is equivalent to the syntactic category $\mathcal{S}_C$ of the simply typed lambda calculus and (2) that $\mathbb{S}_C$, seen as a $\mathbb{B}$-enriched category (where $\mathbb{B}$ is the booleans), is equivalent to the result of changing the base of enrichment of the ($\mathcal{S}et$-enriched) category $\mathcal{S}_C^{\text{nd}}$ along the 'truncation' functor $\tau : \mathcal{S}et \to \mathbb{B}$ given by identifying all elements of a set (one should replace $\mathbb{B}$ with something else if one wants to be more constructive). Here we describe the analogous story in our setting.

Construction 2: Let $\overline{\mathbb{W}} := [\mathbb{W}^{op}, \mathbb{B}]$. Given constants C, let $\mathbb{S}_C$ be the $\overline{\mathbb{W}}$-category whose elements are the types of $\mathbb{W}$-enriched natural deduction with C and where $\hom_{\mathbb{S}_C}(S, T)(w)$ is inhabited if and only if there is a derivation of $\Gamma \vdash_w T$. The rule VAR gives us identities, and the rule SUBST gives composition. ◀

Example 25: When $\mathbb{W} = \mathbb{1}$, then $\overline{\mathbb{W}} = \mathbb{B}$, and we obtain the usual syntactic preordered set of natural deduction. ◀

Construction 3: Given constants C, let $\mathcal{S}_C^{\text{nd}}$ denote the $\widehat{\mathbb{W}}$-category whose objects are the types of $\mathbb{W}$-enriched natural deduction with C and where $\hom_{\mathcal{S}_C^{\text{nd}}}(S,T)(w)$ are derivations of $S \vdash_w T$. Since the collection of objects and each set $\hom_{\mathcal{S}_C^{\text{nd}}}(S,T)(w)$ is in bijection with the analogous collections of sets comprising $\mathcal{S}_C$ (Construction 1), we conclude that this also forms a $\widehat{\mathbb{W}}$-category, equivalent to $\mathcal{S}_C$. ◀

Example 26: When $\mathbb{W} = \mathbb{1}$, we obtain the usual syntactic category of natural deduction. ◀

By the constructions of $\mathcal{S}_C^{\text{nd}}$ and $\mathbb{S}_C$ we find the following.

Proposition 3: Given constants C, $\mathbb{S}_C$ is equivalent to the $\overline{\mathbb{W}}$-category obtained by changing the base of $\mathcal{S}_C^{\text{nd}}$ along the functor $\tau_* : \widehat{\mathcal{W}} \to \overline{\mathcal{W}}$ given by postcomposition with $\tau : \mathcal{S}et \to \mathbb{B}$. ◀

This establishes the existence of the syntactic preordered set, the syntactic category, and their relationship. Note also that any interpretation of the $\mathbb{W}$-enriched simply typed lambda calculus in a $\overline{\mathbb{W}}$-category factors through the interpretation of the $\mathbb{W}$-enriched simply typed lambda calculus in $\mathbb{W}$-enriched natural deduction, producing an interpretation the $\mathbb{W}$-enriched natural deduction in the same $\overline{\mathbb{W}}$-category. Thus, we find the following as a corollary of Theorem 1 and the fact that there is certainly an interpretation into the syntactic $\overline{W}$-category given in Construction 2.

Proposition 4: There is an interpretation of $\mathbb{W}$-enriched natural deduction in any cartesian closed $\overline{W}$-category. This interpretation is sound and complete. ◀

Example 27: When $\mathbb{W} = \mathbb{1}$, this is the usual interpretation of natural deduction in cartesian closed preordered sets. ◀

To slickly expand this interpretation, we employ the change-of-base strategy explained in Remark 2.

Definition 7: For a monoidal posets $\mathbb{W}, \mathbb{V}$, call $\mathbb{V}$ a $\mathbb{W}$-*relative monoidal poset* if as categories, $\mathbb{V}$ is a $\mathbb{W}$-relative monoidal category, i.e., if there is a homomorphism $\iota : \mathbb{W} \to \mathbb{V}$. ◀

Given a $\mathbb{W}$-relative monoidal poset $\mathbb{V}$, we take the restricted Yoneda embedding $よ_\iota : \mathbb{V} \to \overline{\mathbb{W}}$ given by $よ_\iota(v)(w) := \hom_{\mathbb{V}}(\iota w, v)$, where we view $\mathbb{V}$ as a $\mathbb{B}$-enriched category so that this takes values in $\mathbb{B}$. Thus, we give an interpretation of $\mathbb{W}$-natural deduction in any $\mathbb{V}$-enriched category $\mathcal{C}$ by changing the base of $\mathcal{C}$ to obtain a $\overline{\mathbb{W}}$-enriched category and interpreting the syntax there, and we thus obtain the following as a corollary of Theorem 1 and Proposition 4.

Theorem 5: Consider a monoidal poset $\mathbb{W}$ and a $\mathbb{W}$-relative monoidal poset $\mathbb{V}$ with finite meets and which is powered by $\mathbb{W}$.

There is an interpretation of $\mathbb{W}$-enriched natural deduction in any cartesian closed $\mathbb{V}$-category. This interpretation is sound and complete. ◀

C. $\mathbb{W}$-enriched natural deduction as an intuitionistic fuzzy logic

We see $\mathbb{W}$-enriched natural deduction as an intuitionistic fuzzy logic. Fuzzy logic [20] has many instantiations. It is usually presented as a generalization of classical logic, and with that motivation, the perspective taken is then that the formulas in n variables of fuzzy logic, which are built out of connectives similar to those of classical logic, have semantics in functions $\mathbb{W}^n \to \mathbb{W}$ (and usually, $\mathbb{W} := \mathbb{I}$). Thus much of the literature works on extending the truth table semantics of classical logic to the fuzzy setting. A minority of the literature (e.g., [27, 5, 3]) proposes (or works with) an intuitionistic fuzzy logic. Our rules for $\mathbb{W}$-natural deduction present another alternative for an intuitionistic fuzzy logic, one which we propose is a more faithful generalization of intuitionistic logic, at least insofar as it is the only one which generalizes the Curry-Howard-Lambek correspondence that we see as central to intuitionistic logic.

In Construction 2, we gave syntactic $\overline{\mathbb{W}}$-category of $\mathbb{W}$-natural deduction that generalizes the semantics of natural deduction in preordered sets. To make the connection with fuzzy logic tighter, we also want to see an interpretation of these rules into $\mathbb{W}$-categories. That is, in semantics of intuitionistic logic in preordered sets, one can find out if there is a derivation $\Gamma \vdash T$ by calculating $\hom(\Gamma, T) \in \mathbb{B}$. In our setting, we want to have semantics of our fuzzy intuitionistic logic in $\mathbb{W}$-categories (or similar), so that we can find out to what extent there is a derivation $\Gamma \vdash T$ by calculating $\hom(\Gamma, T) \in \mathbb{W}$.

So suppose (as in Section III) that $\mathbb{W}$ has all finite products and is closed. This is a $\mathbb{W}$-relative monoidal category using the identity functor. By Theorem 1, there is an interpretation of the $\mathbb{W}$-enriched simply typed lambda calculus in any $\mathbb{W}$-category $\mathcal{C}$ which factors through an interpretation of $\mathbb{W}$-enriched natural deduction in $\mathcal{C}$.

Now note that something is slightly off: we want to generalize the truth values of logic from $\mathbb{B}$ to $\mathbb{W}$. However, the usual rules of natural deduction coincide with the rules for $\mathbb{1}$-enriched natural deduction, not $\mathbb{B}$-enriched natural deduction, so we have generalized our 'truth values' from $\mathbb{1}$ to $\mathbb{W}$, not from $\mathbb{B}$ to $\mathbb{W}$ (yet). There certainly is an interpretation of $\mathbb{1}$-enriched natural deduction in $\mathbb{1}$-enriched categories, but this is trivial. The booleans $\mathbb{B}$ appear because of the need to have an object of the enriching poset which represents having no derivation $\Gamma \vdash T$. Thus, we interpret $\mathbb{W}$-natural deduction into $\mathbb{W}_0$-categories, where $\mathbb{W}_0$ is $\mathbb{W}$ freely adjoined with an initial, absorbing element 0. Then the inclusion $\iota : \mathbb{W} \hookrightarrow \mathbb{W}_0$ is a homomorphism of monoidal posets, and assuming $\mathbb{W}$ has finite meets and exponentials, then $\mathbb{W}_0$ has finite meets and is powered by $\mathbb{W}$. Thus, by Theorem 5, we obtain an interpretation of $\mathbb{W}$-enriched natural deduction in any cartesian closed $\mathbb{W}_0$-category.

We showed above in Proposition 4 that there is a sound and complete interpretation of $\mathbb{W}$-natural deduction in cartesian closed $\overline{\mathbb{W}}$-categories. Now we show a similar result for $\mathbb{W}_0$-

categories, and this result can be seen as a truncation of Proposition 1.

Proposition 5: Suppose that $\mathbb{W}_0$ is cocomplete. The restricted Yoneda embedding $\text{よ}_\iota : \mathbb{W}_0 \to \overline{\mathbb{W}}$ is an injection, and its image consists of all those $P \in \overline{\mathbb{W}}$ that contain maximal elements (seen as subsets of $\mathbb{W}$). ◀

Proof: The restricted Yoneda embedding has a retraction $P \mapsto \vee P$ and so is an injection. On the sub-preordered set of $\overline{\mathbb{W}}$ described in the statement, this restricts to a bijection. ■

Now, if we add the following proof-irrelevant versions of the rules discussed in Section III-H, we characterize the image of the restricted Yoneda embedding, and thus we obtain a sound and complete interpretation of $\mathbb{W}$-enriched natural deduction with these new rules in any cartesian closed $\mathbb{W}_0$-category.

$$\frac{\Gamma \vdash_w T}{\max(t) \text{ wt}} \qquad \frac{\Gamma \vdash_w T}{w \leq \max(t) \text{ wt}} \qquad \frac{\Gamma \vdash_w T}{\Gamma \vdash_{\max(t)} T}$$

Theorem 6: If $\mathbb{W}$ is cartesian closed and cocomplete, then there is a sound and complete interpretation of $\mathbb{W}$-natural deduction together with the rules above in cartesian closed $\mathbb{W}_0$-categories. ◀

D. Comparison with other intuitionistic fuzzy logics

We end this section with a discussion of other intuitionistic fuzzy logics in the literature.

Takeuti and Titani [27] are usually credited with intuitionistic fuzzy logic. They consider the collection of truth values to be $\mathbb{I}$. They add rules (but no judgments) to Gentzen's system LJ [10] (which is also basically what we have generalized with our $\mathbb{W}$-enriched natural deduction) to obtain a logic in which one can reason about judgments of the form $\Gamma \vdash_w t : T$ only for $w = 1$ (in our notation). They obtain a soundness and completeness result stating that a judgment $\Gamma \vdash_1 t : T$ is derivable if and only if it is valid, meaning that seen as a function $\mathbb{I}^n \to \mathbb{I}$, it is constant at 1. Thus, in comparison with this work, we add judgments and rules to be able to reason about $\Gamma \vdash_w t : T$ for any w, and we obtain soundness and completeness results governing any such judgment. On the other hand, in comparison with our work, they add rules specific to $\mathbb{I}$.

Baaz, Chiabattoni, and Fermüller [5] give an alternate natural deduction system for the logic developed in [27]. But like [27], they only reason syntactically about formulas like $\Gamma \vdash_w t : T$ for $w = 1$.

Atanassov [3] gives an alternate intuitionistic fuzzy logic, though again the truth values lie in $\mathbb{I}$. They see the move from classical fuzzy logic to intuitionistic fuzzy logic as assigning to every proposition, not only a truth value, but a falsity value, an approach that is incomparable to ours.

We see the strength of our approach being that it straightforwardly generalizes not only intuitionistic logic, but also the Curry-Howard-Lambek correspondence that characterizes it, and also that we obtain soundness and completeness results regarding judgments $\Gamma \vdash_w t : T$ for all $w \in \mathbb{W}$.

V. Conclusion and Future Work

In this work, we have defined, for any monoidal category $\mathbb{W}$, (1) the $\mathbb{W}$-enriched simply typed lambda calculus, (2) $\mathbb{W}$-enriched natural deduction, and (3) their semantics in cartesian closed enriched categories, establishing an enriched version of the Curry-Howard-Lambek correspondence and showing that in the Lambek part, the interpretation is sound and complete.

We have given many examples of enriching categories to which this applies, pointing to potential connections with directed type theory and axiomatic domain theory in particular. However, in this paper, we have only explored the examples of fuzzy sets and fuzzy logic, giving sound and complete semantics in those settings by adding characterizing syntactical rules, and comparing our resulting intuitionistic fuzzy logic to others in the literature.

In future work, we hope to develop other examples further, generalize the setting, and develop a dependent version, as detailed below.

We see the need for a coherent mathematical theory that encompasses the current zoo of new modal (or modal-esque) type theories being developed as one of the most important problems in the mathematical study of type theory currently. We hope that the perspective on modal(-esque) type theories described Example 17 will be instructive in the sense that it offers a direction for developing the categorical semantics of at least some of the features offered in these type theories.

We also hope to generalize this work further by, for instance, investigating to what extent the functor $\text{よ}_\iota : \mathcal{V} \to \widehat{\mathcal{W}}$ needs to factor through $\widehat{\mathcal{V}}$ (i.e., to what extent the 'features' we extract need to be representable), and if there are any interesting examples where it does not. Furthermore, could the requirement that ι is strong monoidal be relaxed? These are avenues for modest generalization, but we do not see this work as extending to any arbitrary enriched category. Without a radical change in meaning of the word syntax, the syntax of any logic or type theory always forms a collection of sets. Thus, it is hard to imagine straying very far from the presheaf-central story we have presented here.

The obvious next step, however, is to develop a dependent-typed version, generalizing both our $\mathcal{W}$-enriched simply typed lambda calculus and Martin-Löf type theory. Such a $\mathcal{W}$-enriched dependent type theory will have semantics in (among other structures) an enriched version of display map categories, and it will have type formers corresponding to certain weighted limits, among others.

References

[1] Samson Abramsky and Achim Jung. "Domain theory". In: *Handbook of logic in computer science (vol. 3): semantic structures*. USA: Oxford University Press, Inc., Jan. 1995, pp. 1–168.

[2] Benedikt Ahrens, Paige Randall North, and Niels van der Weide. “Semantics for two-dimensional type theory”. In: *Proceedings of the 37th Annual ACM/IEEE Symposium on Logic in Computer Science*. LICS ’22. New York, NY, USA: Association for Computing Machinery, Aug. 2022, pp. 1–14. DOI: 10.1145/3531130.3533334.

[3] Krassimir T. Atanassov. *Intuitionistic Fuzzy Logics*. Vol. 351. Studies in Fuzziness and Soft Computing. Cham: Springer International Publishing, 2017. DOI: 10.1007/978-3-319-48953-7.

[4] Robert Atkey. “Syntax and Semantics of Quantitative Type Theory”. In: *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*. Oxford United Kingdom: ACM, July 2018, pp. 56–65. DOI: 10.1145/3209108.3209189.

[5] Matthias Baaz, Agata Ciabattoni, and Christian G. Fermüller. “A Natural Deduction System for Intuitionistic Fuzzy Logic”. In: *Lectures on Soft Computing and Fuzzy Logic*. Ed. by Antonio Di Nola and Giangiacomo Gerla. Heidelberg: Physica-Verlag HD, 2001, pp. 1–18. DOI: 10.1007/978-3-7908-1818-5_1.

[6] Kostia Chardonnet, Louis Lemonnier, and Benoît Valiron. “Semantics for a Turing-Complete Reversible Programming Language with Inductive Types”. In: *9th International Conference on Formal Structures for Computation and Deduction (FSCD 2024)*. Ed. by Jakob Rehof. Vol. 299. Leibniz International Proceedings in Informatics (LIPIcs). ISSN: 1868-8969. Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2024, 19:1–19:19. DOI: 10.4230/LIPIcs.FSCD.2024.19.

[7] Alonzo Church. “A formulation of the simple theory of types”. In: *The Journal of Symbolic Logic* 5.2 (June 1940), pp. 56–68. DOI: 10.2307/2266170.

[8] Brian J. Day. “Construction of Biclosed Categories”. PhD thesis. University of New South Wales, Sept. 1970.

[9] Marcelo P. Fiore. *Axiomatic Domain Theory in Categories of Partial Maps*. Distinguished Dissertations in Computer Science. Cambridge: Cambridge University Press, 1996. DOI: 10.1017/CBO9780511526565.

[10] Gerhard Gentzen. “Untersuchungen über das logische Schließen. I”. In: *Mathematische Zeitschrift* 39.1 (Dec. 1935), pp. 176–210. DOI: 10.1007/BF01201353.

[11] Robert Ghrist, Miguel Lopez, Paige Randall North, and Hans Riess. *Categorical Diffusion of Weighted Lattices*. arXiv:2501.03890 [math]. Jan. 2025. DOI: 10.48550/arXiv.2501.03890.

[12] J. A Goguen. “L-fuzzy sets”. In: *Journal of Mathematical Analysis and Applications* 18.1 (Apr. 1967), pp. 145–174. DOI: 10.1016/0022-247X(67)90189-8.

[13] Daniel Gratzer, G. A. Kavvos, Andreas Nuyts, and Lars Birkedal. “Multimodal Dependent Type Theory”. In: *Logical Methods in Computer Science* Volume 17, Issue 3 (July 2021). Publisher: Episciences.org. DOI: 10.46298/lmcs-17(3:11)2021.

[14] Daniel Gratzer, Jonathan Weinberger, and Ulrik Buchholtz. *Directed univalence in simplicial homotopy type theory*. arXiv:2407.09146 [cs]. July 2024. DOI: 10.48550/arXiv.2407.09146.

[15] Robin Kaarsgaard, Holger Bock Axelsen, and Robert Glück. “Join inverse categories and reversible recursion”. In: *Journal of Logical and Algebraic Methods in Programming* 87 (Feb. 2017), pp. 33–50. DOI: 10.1016/j.jlamp.2016.08.003.

[16] G.M. Kelly. *Basic concepts of enriched category theory*. Vol. 64. Lecture Notes in Mathematics. 1984.

[17] Peter Kristel and Benedikt Peterseim. *A Topologically Enriched Probability Monad on the Cartesian Closed Category of CGWH Spaces*. arXiv:2404.08430 [math]. Nov. 2024. DOI: 10.48550/arXiv.2404.08430.

[18] Jean-Simon Pacaud Lemay. “An Ultrametric for Cartesian Differential Categories for Taylor Series Convergence”. In: *Electronic Notes in Theoretical Informatics and Computer Science* Volume 4 - Proceedings of... (Dec. 2024). arXiv:2405.19474 [math], p. 14670. DOI: 10.46298/entics.14670.

[19] Daniel R. Licata, Michael Shulman, and Mitchell Riley. “A Fibrational Framework for Substructural and Modal Logics”. In: *2nd International Conference on Formal Structures for Computation and Deduction (FSCD 2017)*. Ed. by Dale Miller. Vol. 84. Leibniz International Proceedings in Informatics (LIPIcs). ISSN: 1868-8969. Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2017, 25:1–25:22. DOI: 10.4230/LIPIcs.FSCD.2017.25.

[20] Jan Łukasiewicz and Alfred Tarski. “Untersuchungen uber den Aussagenkalkul”. In: *Comptes Rendus Des Séances de La Société Des Sciences et Des Lettres de Varsovie, Cl. III* 23.iii (1930).

[21] Yue Niu, Jonathan Sterling, and Robert Harper. “Cost-sensitive computational adequacy of higher-order recursion in synthetic domain theory”. In: *Electronic Notes in Theoretical Informatics and Computer Science* Volume 4 - Proceedings of... (Dec. 2024). arXiv:2404.00212 [cs], p. 14732. DOI: 10.46298/entics.14732.

[22] Paige Randall North. “Towards a Directed Homotopy Type Theory”. In: *Electronic Notes in Theoretical Computer Science*. Proceedings of the Thirty-Fifth Conference on the Mathematical Foundations of Programming Semantics 347 (Nov. 2019), pp. 223–239. DOI: 10.1016/j.entcs.2019.09.012.

[23] Andreas Nuyts. “Towards a Directed Homotopy Type Theory based on 4 Kinds of Variance.(2015)”. PhD Thesis. KU Leuven, 2015.

[24] Paolo Perrone and Ruben Van Belle. *Convergence of martingales via enriched dagger categories*. arXiv:2404.15191 [math]. Apr. 2024. DOI: 10.48550/arXiv.2404.15191.

[25] Cécilia Pradic and Ian Price. “Implicit automata in λ-calculi III: affine planar string-to-string functions”. In: *Electronic Notes in Theoretical Informatics and

Computer Science 4 (Dec. 2024). arXiv:2404.03985 [cs], p. 14804. DOI: 10.46298/entics.14804.

[26] Philip Saville. "Clones, closed categories, and combinatory logic". In: *Foundations of Software Science and Computation Structures*. Ed. by Naoki Kobayashi and James Worrell. Cham: Springer Nature Switzerland, 2024, pp. 160–181. DOI: 10.1007/978-3-031-57231-9_8.

[27] Gaisi Takeuti and Satoko Titani. "Intuitionistic Fuzzy Logic and Intuitionistic Fuzzy Set Theory". In: *The Journal of Symbolic Logic* 49.3 (1984). Publisher: [Association for Symbolic Logic, Cambridge University Press], pp. 851–866. DOI: 10.2307/2274139.

[28] Takeshi Tsukada and Kazuyuki Asada. "Enriched Presheaf Model of Quantum FPC". In: *Proc. ACM Program. Lang.* 8.POPL (Jan. 2024), 13:362–13:392. DOI: 10.1145/3632855.

[29] L. A. Zadeh. "Fuzzy sets". In: *Information and Control* 8.3 (June 1965), pp. 338–353. DOI: 10.1016/S0019-9958(65)90241-X.