A Higher Structure Identity Principle

Extended abstract

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Abstract

The ordinary Structure Identity Principle states that any property of set-level structures (e.g., posets, groups, rings, fields) that is definable in Univalent Foundations is invariant under isomorphism: more specifically, the identity type between such structures coincides with the type of isomorphisms. We prove a version of this principle applicable to a wide range of higher-categorical structures, adapting FOLDS-signatures to specify a general class of such structures, and using two-level type theory to treat all categorical dimensions uniformly. As in the previously known case of 1-categories (which is an instance of our theory), to make this true the structures themselves must satisfy a local form of the univalence principle, stating that the identity type between any two elements of the structure coincides with a type of "isomorphisms" between them. Our main technical achievement is to give a general definition of such isomorphisms between elements of any higher-dimensional structure that can be specified by a signature.

Keywords homotopy type theory, univalent foundations, structure identity principle, categories, equivalence principle

1 Introduction

A fundamental logical principle is the "indiscernibility of identicals", i.e., equal objects have the same properties:

$$x = y \to \forall \text{ properties } P, (P(x) \leftrightarrow P(y))\,. \tag{1}$$

However, properties invariant under weaker notions of sameness are also important. For instance, group-theoretic properties satisfy a similar principle for *isomorphic* groups:

$$G \cong H \to \forall \text{ group-theoretic properties } P, (P(G) \leftrightarrow P(H))\,;$$

while category-theoretic properties are invariant even under *equivalence* of categories:

$$A \simeq B \to \forall \text{ category-theoretic properties } P, (P(A) \leftrightarrow P(B))\,.$$

The idea is summarized in Aczel's Structure Identity Principle (SIP) [1]: *Isomorphic mathematical structures are structurally identical; i.e., have the same structural properties.* But it remains to characterize the "structural properties" for a given kind of mathematical structure. For instance, Blanc [5] and Freyd [8] devised a syntax for category-theoretic properties and showed that they are invariant under equivalence.

Makkai [10] introduced a notion of signature for higher-categorical structures, along with a notion of equivalence and a language for properties of such structures, called First Order Logic with Dependent Sorts (FOLDS), and proved that FOLDS-properties are invariant under FOLDS-equivalence.

1.1 Structure Identity Principle in Univalent Foundations

Inspired by Makkai (see [12, p. 1279]), Voevodsky conceived Univalent Foundations (UF) with a similar goal: a language whose constructions can be mechanically transported along equivalences of structures. Since proofs are particular constructions, this implies a similar invariance of properties, as considered by Makkai, Blanc, and Freyd.

The formal language of UF and the closely related Homotopy Type Theory (HoTT) is Martin-Löf type theory, with types regarded as (higher) groupoids; see [11] for background and notation. The invariance is achieved by combining the "transport" function for the Martin-Löf identity type

$$\Pi_{(x,y:\mathcal{U})} \left(x =_{\mathcal{U}} y \to \Pi_{(P:\mathcal{U}\to\mathcal{U})} \left(P(x) \simeq P(y)\right)\right). \tag{2}$$

with Voevodsky's univalence principle

$$\mathsf{univalence} : \Pi_{(x,y:\mathcal{U})}(x =_{\mathcal{U}} y \xrightarrow{\sim} x \simeq y)$$

that replaces the hypothesis $x = y$ in Equation (2) by an equivalence of types $x \simeq y$ (see also [4]):

$$\Pi_{(x,y:\mathcal{U})} \left(x \simeq y \to \Pi_{(P:\mathcal{U}\to\mathcal{U})} \left(P(x) \simeq P(y)\right)\right).$$

Based on this transport principle for types, it was proven by [11, Section 9.9] and independently by [7] (the formalization accompanying [7] compares the two) that for a wide range of mathematical structures, *any* property expressible in HoTT/UF is "structural" as in Aczel's SIP. Both use a notion of "signature" to define general classes of structures and isomorphisms, and show that such isomorphisms are equivalent to identifications, obtaining a transport principle:

$$\Pi_{(x,y:\mathsf{Str}(S))} \left(x \cong_S y \to \Pi_{(P:\mathsf{Str}(S)\to\mathcal{U})} \left(P(x) \simeq P(y)\right)\right).$$

Here, think of x and y as "sets equipped with structure"; instances include posets, monoids, groups, and fields.

1.2 An SIP for categories

The main restriction of the SIP of Section 1.1 is that it mostly applies to structures that naturally form a 1-category (or rather, a 1-groupoid). In particular, it excludes properties of categories themselves. But an SIP for categories is proved in [2, Theorem 6.17]: equivalence of "univalent" categories is equivalent to identity of categories, yielding an analogous transport principle for categorical equivalences $x \simeq y$:

$$\Pi_{(x,y:\mathsf{Cat})} \left(x \simeq y \to \Pi_{(P:\mathsf{Cat}\to\mathcal{U})} \left(P(x) \simeq P(y)\right)\right).$$

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We review the details in Section 2.

In the present paper, grown out of [?], we generalize this to other (higher-)categorical structures. We define (higher) structures via suitable signatures, univalence of structures, and equivalence of structures, such that for any univalent $\mathcal{L}$-structures x, y, properties and constructions can be transported along equivalences:

$$\Pi_{(x,y:\mathrm{uStr}(\mathcal{L}))}\left(x \simeq_{\mathcal{L}} y \to \Pi_{(P:\mathrm{uStr}(\mathcal{L})\to\mathcal{U})}\left(P(x) \simeq P(y)\right)\right).$$

It turns out that the SIP of Section 1.1 is equivalent to the statement that the 1-category *of* structures is univalent in this sense. Thus there appears to be a "Baez–Dolan microcosm principle" at work: the SIP *for* a given kind of structure states that the (higher) category *of* such structures satisfies the necessary univalence principle for *its* SIP.

2 A fresh look at univalent categories

In [2] and [11, Chapter 9], category theory in HoTT/UF began by defining a **precategory** C to consist of the following.

- A type C_0 of objects.
- For each $a, b : C_0$, a set $C(a, b)$ of morphisms.
- For each $a : C_0$, a morphism $1_a : C(a, a)$.
- For each $a, b, c : C_0$, a function
$$C(b, c) \to C(a, b) \to C(a, c).$$
- For each $a, b : C_0$ and $f : C(a, b)$, we have $f = 1_b \circ f$ and $f = f \circ 1_a$.
- For each $a, b, c, d : C_0$ and $f : C(a, b)$, $g : C(b, c)$, $h : C(c, d)$, we have $h \circ (g \circ f) = (h \circ g) \circ f$.

Note that C_0 may not be a set, and in practice for "large" precategories it almost never is. For instance, Set_0 is the type of sets, which by univalence is a proper 1-type. However, allowing arbitrary types of objects is problematic too. For instance, while the statement "a fully faithful and essentially surjective functor is an equivalence" in ZFC is equivalent to the axiom of choice, for precategories in HoTT/UF it is simply *false*.

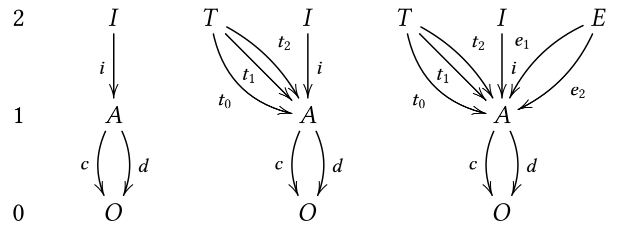
The solution is to impose a "local univalence" condition: for any $a, b : C_0$ there is a map $\mathrm{idtoiso}_{a,b} : (a =_{C_0} b) \to (a \cong b)$, defined by path induction, and a precategory C is a **univalent category** if $\mathrm{idtoiso}_{a,b}$ is an equivalence for all $a, b : C_0$. This implements the idea that "isomorphic objects are equal". Note that it implies that C_0 is a 1-type, since its identity types are all sets (0-types).

One then proves, using the univalence axiom, that this "local" form of univalence for objects of a category implies a "global" form of univalence for categories themselves:

Theorem 2.1 ([2, Theorem 6.17]). *For univalent categories C and $\mathcal{D}$, let $C \simeq \mathcal{D}$ consist of equivalence functors; then*

$$(C = \mathcal{D}) = (C \simeq \mathcal{D}).$$

We will generalize this to other categorical structures, starting with a general vocabulary for expressing such things.



$$ci \overset{s}{=} di, ct_0 \overset{s}{=} dt_1, dt_0 \overset{s}{=} dt_2, ct_1 \overset{s}{=} ct_2, de_1 \overset{s}{=} de_2, ce_1 \overset{s}{=} ce_2$$

Figure 1. The signatures $\mathcal{L}_{\mathrm{rg}}$, $\mathcal{L}_{\mathrm{cat}}$, and $\mathcal{L}_{\mathrm{cat+E}}$ (from left to right) for reflexive graphs, for categories, and for categories with equality predicate on arrows.

2.1 FOLDS-signature for categories

In [10], Makkai presents a definition of category in a language called First-Order Logic with Dependent Sorts (FOLDS). In contrast to HoTT/UF, FOLDS is not a foundational system for mathematics, but a kind of first-order logic designed for higher categorical structures. We will not use the logical syntax of FOLDS, but we adopt and generalize its semantic notions of signature and structure.

A FOLDS-signature is a *one-way category*, whose objects we call *sorts*. The FOLDS-signature $\mathcal{L}_{\mathrm{cat}}$ of categories is shown in Figure 1, along with the related signatures $\mathcal{L}_{\mathrm{rg}}$ of reflexive graphs and $\mathcal{L}_{\mathrm{cat+E}}$ of categories with equality (see Section 2.2). There are some relations on the composite arrows (e.g., the two composites $I \to A \rightrightarrows O$ are equal). The intent is that O is the sort of objects, A the sort of arrows, I the sort of identity arrows, and T the sort of composable pairs of arrows (with their composite; T stands for "triangle").

Using a set-theoretic metatheory, Makkai defined *structures* for a FOLDS-signature as certain functors into **Set**. In a dependently typed theory like HoTT/UF, however, it is more natural to interpret each sort as a *dependent* type indexed by the interpretations of all the sorts below it. For instance, a structure M for the signature $\mathcal{L}_{\mathrm{cat}}$ in Figure 1 consists of

$$\begin{aligned}
MO &: \mathcal{U}\\
MA &: MO \times MO \to \mathcal{U}\\
MI &: \textstyle\prod_{(x:MO)} MA(x,x) \to \mathcal{U}\\
MT &: \textstyle\prod_{(x,y,z:MO)} MA(x,y) \to MA(y,z) \to MA(x,z) \to \mathcal{U}
\end{aligned}$$

This forms the underlying data of a category: a type of objects, types of morphisms, and relations of "being an identity" and "being the composite". Later we will see that our univalence condition implies that the families MI and MT consist of propositions and MA consists of sets.

2.2 Equations and theories

To express which such structures are actually categories, Makkai introduced a logic over FOLDS-signatures, taking top-level sorts as "relation symbols". For instance, the axiom that any two composable arrows have a composite would be

$$\forall(x,y,z:O).\forall(f:A(x,y)).\forall(g:A(y,z)).\\ \exists(h:A(x,z)).T_{x,y,z}(f,g,h).$$

Any such axiom can be interpretated as a predicate on $\mathcal{L}$-structures in HoTT/UF. (Note that the interpretation of $\exists$ and $\vee$ involves propositional truncation.)

The axioms of a category also involve equality of arrows (though not of objects), e.g., the uniqueness of composites

$$\forall(x,y,z:O).\forall(f:A(x,y)).\forall(g:A(y,z)).\forall(h,h':A(x,z)).\\ T_{x,y,z}(f,g,h)\to T_{x,y,z}(f,g,h')\to(h=h').$$

Just as in ordinary first-order logic, one can consider FOLDS either *with equality* or *without equality*. In the former, equality is only allowed between elements of sorts that are "one level below the top" like A. As usual, FOLDS with equality can be embedded in FOLDS without equality by adding equality relations to the signature, as with $\mathcal{L}_{\text{cat+E}}$ in Figure 1, along with axioms making them congruences:

$$E_{x,y}(f,f)$$
$$E_{x,y}(f,g)\to E_{x,y}(g,f)$$
$$E_{x,y}(f,g)\wedge E_{x,y}(g,h)\to E_{x,y}(f,h)$$
$$E_{x,x}(f,g)\wedge I_x(f)\to I_x(g)$$
$$E_{x,y}(f,f')\wedge E_{y,z}(g,g')\wedge E_{x,z}(h,h')\wedge T_{x,y,z}(f,g,h)\\ \to T_{x,y,z}(f',g',h').$$

A model M is **standard** if $ME_{x,y}$ is equivalent to the actual equality of $MA(x,y)$. Like truncatedness, this will turn out to be a special case of our univalence condition.

Remark 2.2. We do not use, in the present work, a formal account of axioms and theories. We have only mentioned them informally to compare, as a sanity check, our FOLDS-categories to the categories of [2].

2.3 FOLDS-categories in univalent foundations

As noted above, in HoTT/UF we must consider what truncation level MO and $MA(x,y)$ should have. In a precategory, we require the types of arrows to be sets, suggesting the following analogous definition.

Definition 2.3. A **1-univalent FOLDS-category** M consists of

- A type $MO:\mathcal{U}$;
- A family $MA:MO\times MO\to\mathcal{U}$;
- A family $MI:\prod_{(x:MO)}MA(x,x)\to\mathcal{U}$;
- A family $MT:\prod_{(x,y,z:MO)}MA(x,y)\to MA(y,z)\to MA(x,z)\to\mathcal{U}$; and
- A family $ME:\prod_{(x,y:MO)}MA(x,y)\to MA(x,y)\to\mathcal{U}$,

such that

- Each type $MI_x(f)$, $MT_{x,y,z}(f,g,h)$, and $ME_{x,y}(f,g)$ is a proposition;
- Each type $MA(x,y)$ is a set,
- $ME_{x,y}(f,g)\leftrightarrow(f=g)$,

and the axioms of a category are satisfied.

Lemma 2.4. *The type of 1-univalent FOLDS-categories is equivalent to the type of precategories.*

Convention 2.5. Below, we sometimes abuse notation by writing $x:O$ instead of $x:MO$, and similarly for the other sorts, when the particular structure M is clear from context.

Let us now consider how to define "univalent categories" using only the FOLDS-structure. The central problem is to characterize the type $(a\cong b)$ of isomorphisms without using our knowledge that these data describe a category.

To start with, recall that by the Yoneda lemma, an isomorphism $\phi:a\cong b$ in a category C is equivalently a natural family of isomorphisms of sets $\phi_{x\bullet}:C(x,a)\cong C(x,b)$, where naturality in x means that $\phi_{y\bullet}(g)\circ f=\phi_{x\bullet}(g\circ f)$. In the language of FOLDS-categories the operation $\circ$ is replaced by the relation T, with a new variable h for the composite $g\circ f$:

- For each $x:O$, an isomorphism $\phi_{x\bullet}:A(x,a)\cong A(x,b)$.
- For each $x,y:O$, $f:A(x,y)$, $g:A(y,a)$, and $h:A(x,a)$, we have $T_{x,y,a}(f,g,h)\leftrightarrow T_{x,y,b}(f,\phi_{y\bullet}(g),\phi_{x\bullet}(h))$.

This looks more promising, but it still privileges one of the variables of A over the other, and the relation T over I (and E). More natural from the FOLDS point of view is to give equivalences between hom-sets with a and b substituted into *all* possible "collections of holes":

1. For each $x:O$, an isomorphism $\phi_{x\bullet}:A(x,a)\cong A(x,b)$.
2. For each $z:O$, an isomorphism $\phi_{\bullet z}:A(a,z)\cong A(b,z)$.
3. An isomorphism $\phi_{\bullet\bullet}:A(a,a)\cong A(b,b)$.

and similar logical equivalences between all possible "relations with holes":

$$T_{x,y,a}(f,g,h)\leftrightarrow T_{x,y,b}(f,\phi_{y\bullet}(g),\phi_{x\bullet}(h)) \tag{3}$$
$$T_{x,a,z}(f,g,h)\leftrightarrow T_{x,b,z}(\phi_{x\bullet}(f),\phi_{\bullet z}(g),h) \tag{4}$$
$$T_{a,z,w}(f,g,h)\leftrightarrow T_{b,z,w}(\phi_{\bullet z}(f),g,\phi_{\bullet w}(h)) \tag{5}$$
$$T_{x,a,a}(f,g,h)\leftrightarrow T_{x,b,b}(\phi_{x\bullet}(f),\phi_{\bullet\bullet}(g),\phi_{x\bullet}(h)) \tag{6}$$
$$T_{a,x,a}(f,g,h)\leftrightarrow T_{b,x,b}(\phi_{\bullet x}(f),\phi_{x\bullet}(g),\phi_{\bullet\bullet}(h)) \tag{7}$$
$$T_{a,a,x}(f,g,h)\leftrightarrow T_{b,b,x}(\phi_{\bullet\bullet}(f),\phi_{\bullet x}(g),\phi_{\bullet x}(h)) \tag{8}$$
$$T_{a,a,a}(f,g,h)\leftrightarrow T_{b,b,b}(\phi_{\bullet\bullet}(f),\phi_{\bullet\bullet}(g),\phi_{\bullet\bullet}(h)) \tag{9}$$
$$I_{a,a}(f)\leftrightarrow I_{b,b}(\phi_{\bullet\bullet}(f)) \tag{10}$$

for all $x,y,z,w:O$ and f,g,h of appropriate types. Fortunately, the additional data here are redundant. Just as (3) means the $\phi_{x\bullet}$ form a natural isomorphism, (5) means the $\phi_{\bullet z}$ form a natural isomorphism, and (4) means these natural isomorphisms arise from the same $\phi:a\cong b$. Given this, any one of Eqs. (6) to (8) ensures that $\phi_{\bullet\bullet}$ is conjugation by ϕ, and then the other two follow automatically, as do Eqs. (9) and (10). This suggests the following definition.

Definition 2.6. For a,b objects of a 1-univalent FOLDS-category, a **FOLDS-isomorphism** from a to b consists of data as in Items 1 to 3 satisfying Eqs. (3) to (10).

Theorem 2.7. *In any* 1*-univalent FOLDS-category, the type of FOLDS-isomorphisms from a to b is equivalent to the type of ordinary isomorphisms $a \cong b$.* □

Definition 2.8. A 0**-univalent FOLDS-category** is a 1-univalent FOLDS-category such that for all $a, b : MO$, the canonical map from $(a = b)$ to the type $(a \cong b)$ of FOLDS-isomorphisms is an equivalence.

Theorem 2.9. *A 1-univalent FOLDS-category is* 0*-univalent iff its corresponding precategory is a univalent category.*

The point is that the definition of FOLDS-isomorphism can be derived algorithmically from the FOLDS-signature for categories, by an algorithm which applies equally well to any FOLDS-signature. We will give this mechanism explicitly in Section 6. Then, for any $a, b : MK$ in some structure M, there will be a canonical map $(a =_K b) \to (a \cong b)$, and we call M **univalent** if these are equivalences.

However, there are two mismatches between this example so far and the general theory we have just proposed. Firstly, we have assumed *ad hoc* that MA consists of sets. Secondly, we have just proposed that *all* sorts should satisfy a univalence property, but in the example of categories we have only considered this for the sort O. Fortunately, these two problems solve each other, and moreover remove the need to postulate "standardness" of equality.

Definition 2.10. A 2**-univalent FOLDS-category** M consists of the same type families MO, MA, MI, MT, ME as a 1-univalent FOLDS-category, such that ME is a congruence, each type $MI_x(f)$, $MT_{x,y,z}(f, g, h)$, and $ME_{x,y}(f, g)$ is a proposition, and the axioms of a category are satisfied with ME used in place of equality.

A FOLDS-isomorphism between $f, g : A(a, b)$ in a 2-univalent FOLDS-category should consist of logical equivalences between instances of T, I, and E with f replaced by g in "all possible ways", clearly beginning with

$$T_{x,a,b}(u, f, v) \leftrightarrow T_{x,a,b}(u, g, v) \tag{11}$$
$$T_{a,x,b}(u, v, f) \leftrightarrow T_{a,x,b}(u, v, g) \tag{12}$$
$$T_{a,b,x}(f, u, v) \leftrightarrow T_{x,a,b}(g, u, v) \tag{13}$$

for all $x : O$ and u, v of appropriate types. But how do we put f in two or three of the places in T in the most general way? In Section 6 we will see that the answer is to assume an equality between objects and transport f along it.

Definition 2.11. For $f, g : A(x, y)$ in a 2-univalent FOLDS-category, a **FOLDS-isomorphism** from f to g consists of the logical equivalences shown in Eqs. (11) to (21), for all $p : a = a$, $q : b = a$, and $r : b = b$.

$$T_{a,a,b}(q_*(f), f, u) \leftrightarrow T_{a,a,b}(q_*(g), g, u) \tag{14}$$
$$T_{a,b,b}(p_*(f), u, f) \leftrightarrow T_{a,b,b}(p_*(g), u, g) \tag{15}$$
$$T_{a,a,b}(u, r_*(f), f) \leftrightarrow T_{a,a,b}(u, r_*(g), g) \tag{16}$$
$$T_{a,a,b}((p,q)_*(f), r_*(f), f) \leftrightarrow T_{a,a,b}((p,q)_*(g), r_*(g), g) \tag{17}$$
$$I_a(q_*(f)) \leftrightarrow I_a(q_*(g)) \tag{18}$$
$$E_{a,b}(f, u) \leftrightarrow E_{a,b}(g, u) \tag{19}$$
$$E_{a,b}(u, f) \leftrightarrow E_{a,b}(u, g) \tag{20}$$
$$E_{a,b}((p,r)_*(f), f) \leftrightarrow E_{a,b}((p,r)_*(g), g) \tag{21}$$

Since T, I, and E are propositions, so is the type $f \cong g$ of FOLDS-isomorphisms. And $f \cong f$, so by path induction we have $(f = g) \to (f \cong g)$.

Theorem 2.12. *A* 2*-univalent FOLDS-category is* 1*-univalent iff the map $(f = g) \to (f \cong g)$ is an equivalence for all f, g.*

Thus, by extending the "univalence" condition of a category from the sort O to the sort A, we encompass automatically the assumption that the hom-types in a precategory are sets and that the equality is standard.

Finally, we can even stop treating the top sorts specially.

Definition 2.13. A **FOLDS-category** consists of the same data and axioms as a 2-univalent FOLDS-category, but without the assumption that the types T, I, and E are propositions.

We can now ask about the type $t \cong t'$ of FOLDS-isomorphisms between two inhabitants $t, t' : T_{x,y,z}(f, g, h)$, say. This should consist of consistent equivalences between all types dependent on t and t'. But there are no such types in the signature, so there is a unique such FOLDS-isomorphism, i.e., $(t \cong t')$ is contractible. The same reasoning applies to I and E. Thus, the univalence condition for these types will assert simply that all of their path-types are contractible, i.e., that they are propositions.

Theorem 2.14. *A FOLDS-category is* 2*-univalent if and only if the following canonical maps are equivalences*

$$(t = t') \to (t \cong t')$$
$$(i = i') \to (i \cong i')$$
$$(e = e') \to (e \cong e')$$

for all inhabitants of the types T, I, and E respectively. □

Thus, the notion of univalent category is determined by only the signature $\mathcal{L}_{\text{cat+E}}$, plus the appropriate axioms (which are irrelevant for defining FOLDS-isomorphisms and univalence). Our goal in this paper will be to define notions of FOLDS-isomorphism and univalence for any signature $\mathcal{L}$, generalizing the theory of univalent categories to arbitrary higher categorical structures.

3 Background: Two-level type theory

In the following sections, we work in a two-level type theory (2LTT) as presented in [3], with axioms (M2) (Russell-style universes), (T1), (T2), and (T3) from [3, Section 2.4]. Building on [9], this theory is shown in [3, §2.5] to be modeled by simplicial sets and Kan complexes.

2LTT consists of an "outer" (also called "strict") level, a form of Martin-Löf type theory with intensional identity types and the principle of uniqueness of identity proofs (UIP), and an "inner" level, a form of homotopy type theory with univalent universes. Both levels come with their own—prima facie distinct—type formers Π, Σ, $+$, $\mathbf{1}$, $\mathbf{0}$, $\mathbb{N}$, intensional "$=$" satisfying function extensionality, and a hierarchy of universes. By axioms (T1) and (T2), we can **identify inner types with particular outer ones** in such a way that the type formers Π, Σ, and $\mathbf{1}$ are "shared" between the levels [3, Lemma 2.11], so we need not distinguish those notationally. For other type constructors we annotate the outer variants with an exponent ^{s} (for "strict"), e.g., in $\mathbb{N}^s$. (Note that in [3], the *inner* type formers are annotated.) We use the conventional typical ambiguity [11, Section 1.3] and hence refer to any universe by $\mathcal{U}$ (inner) resp. $\mathcal{U}^s$ (outer). We use $\equiv$ and $:\equiv$ to denote judgmental equality, e.g., in definitions.

A type being inner is a meta-theoretic statement, but we can internalize it by calling a type A **fibrant** when it is isomorphic to an inner type A' (in the strict sense, modulo $\overset{s}{=}$). Fibrancy is structure rather than property, but following [3] we abuse language by talking about a type "being fibrant" for simplicity. Axiom (T3) states that **every fibrant type is in fact itself inner**. Thus, fibrant types are closed under Π and Σ ([3, Lemma 3.5]). A fibrant type A is equipped with two identity types: for $a, b : A$ we have the strict identity type $a \overset{s}{=} b$ that satisfies UIP, and the homotopical identity type $a = b$ that is at the center of HoTT. For emphasis, we refer, in the following, to elements of the homotopical identity type as "**identifications**", and to elements of the strict identity type to "**strict equalities**". Importantly, since $a = b$ is a fibrant type, it only eliminates into fibrant types; whereas from $a \overset{s}{=} b$ one can eliminate into any type, fibrant or not. Consequently, for any fibrant type A and $a, b : A$, we have a map $a \overset{s}{=} b \to a = b$ from strict equalities to identifications. We sometimes use this map implicitly to "coerce" a strict equality to an identification. Similarly, we will frequently prove statements by induction on the strict natural numbers $\mathbb{N}^s$; there, we do not need to pay attention to the return type. We write $A : \mathcal{U}$ to indicate that A is a fibrant, or inner, type.

We write $A \simeq B$ for the type of equivalences between two (necessarily fibrant) types A to B, in the usual sense of HoTT/UF. The *truncation level* of a fibrant type is defined as in [11], with contractible types the (-2)-types, and an n-type being a type whose homotopical identity types are $(n-1)$-types. A *proposition* is a (-1)-type, and a *set* is a 0-type.

An *s*-**category** C (see also [3, Definition 3.1]) is given by the following data ("s" for "strict"):

1. A type C_0 of *objects* (also often denoted C).
2. For each $x, y : C$ a type $C(x, y)$ of *arrows*.
3. For each $x : C$ an arrow $1 : C(x, x)$.
4. A *composition* map $\circ : C(y, z) \to C(x, y) \to C(x, z)$ that is strictly associative and for which 1 is a strict left and right unit.

A universe $\mathcal{U}^s$ gives rise to an s-category, also called $\mathcal{U}^s$, with objects $A : \mathcal{U}^s$ and morphisms $\mathcal{U}^s(A, B) :\equiv A \to B$. An *s*-functor $F : C \to \mathcal{D}$ consists of a function $F_0 : C_0 \to \mathcal{D}_0$ and functions $F_{x,y} : C(x, y) \to \mathcal{D}(F_0 x, F_0 y)$ preserving identity and composition up to strict equality. We denote both F_0 and $F_{x,y}$ by just F. A strict natural transformation $\alpha : F \Rightarrow G : C \to \mathcal{D}$ consists of a family of morphisms $(\alpha_x : \mathcal{D}(Fx, Gx))_{x:C_0}$ satisfying the naturality axiom strictly.

Our signatures will involve strict equality, and will thus live in the outer level of 2LTT. For a fixed signature, the types of structures, of maps between structures, and of isomorphisms within a structure, will live entirely within the fibrant fragment of 2LTT.

4 Signatures and structures

In traditional logic, a *signature* specifies the sorts, functions, and relations of a structure. A signature in dependent type theory must also specify the dependencies between sorts; Makkai [10] observed that this enables relations and, to a certain extent, functions, to be expressed merely in terms of sorts. Thus we could adapt Makkai's *one-way categories* to 2LTT to obtain a notion of *FOLDS-signatures* and define structures as s-functors to $\mathcal{U}$ that are "Reedy fibrant". However, it will be more convenient to formulate (and generalize) the notion of signature *inductively*.

Consider the FOLDS-signature $\mathcal{L}_{\mathrm{rg}}$, for which a naïve structure consists of (fibrant) types and families $MO : \mathcal{U}$, $MA : MO \to MO \to \mathcal{U}$, and $MI : \prod_{(x:MO)} MA(x, x) \to \mathcal{U}$. If we strip off the *top* sort I, the resulting structure contains only MO and MA, and an inductive definition can be formulated along these lines. But our inductions will be "bottom-up", so we want to strip off the *bottom* sort O. Once $MO : \mathcal{U}$ is fixed, the rest of an $\mathcal{L}_{\mathrm{rg}}$-structure is determined by an ordinary structure over a *derived* signature $(\mathcal{L}_{\mathrm{rg}})'_{MO}$, with a rank-0 sort $A(x, y)$ for each $x, y : MO$, and a rank-1 sort $I(x)$ for each $x : MO$, with morphisms $I(x) \to A(x, x)$. That is, we take the "indexing" of all sorts by O and move it "outside" the signature, incorporating it into the types of sorts.[1]

This notion of *derived FOLDS-signature* determines the notion of structure: a structure for $\mathcal{L}$ of height $p > 0$ consists, inductively, of a family $M_\perp : \mathcal{L}(0) \to \mathcal{U}$ and a structure for $\mathcal{L}'_{M_\perp}$ (which is of height $p - 1$). We can therefore abstract away from one-way categories, remembering only that each signature $\mathcal{L}$ of height $p > 0$ has (1) a type $\mathcal{L}(0)$, and (2) for any $M_\perp : \mathcal{L}(0) \to \mathcal{U}$, a signature $\mathcal{L}'_{M_\perp}$ of height $p - 1$.

Definition 4.1 (Axiomatic signatures). We define a family of s-categories $\mathrm{Sig}(n)$ of signatures of height n by induction. Let $\mathrm{Sig}(0)$ be the trivial s-category on $\mathbf{1}$.

[1] This would be impossible if our one-way categories were metatheoretic in the ordinary sense, e.g., syntactic and externally finite. 2LTT is just right.

An object $\mathcal{L}$ of $\mathsf{Sig}(n+1)$ consists of

1. a fibrant type $\mathcal{L}_\perp : \mathcal{U}$;
2. a functor $\mathcal{L}' : (\mathcal{L}_\perp \to \mathcal{U}) \to \mathsf{Sig}(n)$, where $\mathcal{L}_\perp \to \mathcal{U}$ is the functor s-category from the discrete s-category $L_\perp$ to the canonical s-category $\mathcal{U}$.

Arguments of $\mathcal{L}'$ will be written as subscripts, as in $\mathcal{L}'_M$.

For $\mathcal{L}, \mathcal{M} : \mathsf{Sig}(n+1)$, an element α of $\hom_{\mathsf{Sig}(n+1)}(\mathcal{L}, \mathcal{M})$ consists of the following:

1. a function $\alpha_\perp : \mathcal{L}_\perp \to \mathcal{M}_\perp$
2. a strict natural transformation α' as in the diagram

$$\begin{tikzcd}
\mathcal{M}_\perp \to \mathcal{U} \arrow[d, "_\circ\alpha_\perp"'] \arrow[drr, "\mathcal{M}'"] & & \\
& \arrow[r, phantom, "\alpha'\Uparrow"] & \mathsf{Sig}(n) \\
\mathcal{L}_\perp \to \mathcal{U} \arrow[urr, "\mathcal{L}'"'] & &
\end{tikzcd}$$

Arguments of α' will also be written as subscripts, as in α'_M.

Composition and identities are given by function composition and identity at $\perp$, and inductively for the derivative. Similarly, the categorical laws are easily proved by induction.

Similarly, we define $\mathcal{L}$-*structures* inductively for $n : \mathbb{N}^s$ and each $L : \mathsf{Sig}(n)$. The rank-0 part of an $\mathcal{L}$-structure is a type family $M_\perp : \mathcal{L}_\perp \to \mathcal{U}$, while the rest of an $\mathcal{L}$-structure consists of a structure for the derived signature $\mathcal{L}'_{M_\perp}$.

Definition 4.2 ($\mathcal{L}$**-structures).** If $\mathcal{L} : \mathsf{Sig}(0)$, we define the type of $\mathcal{L}$-*structures* to be $\mathsf{Str}(\mathcal{L}) :\equiv \mathbf{1}$.

If $\mathcal{L} : \mathsf{Sig}(n+1)$, we define the type of $\mathcal{L}$-*structures* to be

$$\mathsf{Str}(\mathcal{L}) :\equiv \sum_{M_\perp : \mathcal{L}_\perp \to \mathcal{U}} \mathsf{Str}(\mathcal{L}'_{M_\perp}).$$

For $\mathcal{L} : \mathsf{Sig}(n+1)$ we write $M :\equiv (M_\perp, M') : \mathsf{Str}(\mathcal{L})$.

Lemma 4.3. *For any signature $\mathcal{L}$, the type $\mathsf{Str}(\mathcal{L})$ of $\mathcal{L}$-structures is fibrant.* □

In the rest of the paper we work exclusively with axiomatic signatures, calling them simply "signatures". However, since most intended examples arise naturally as FOLDS-signatures, we need to be able to translate FOLDS-signatures to axiomatic ones. It is possible to define general FOLDS-signatures in 2LTT and translate them to axiomatic signatures, in such a way that "Reedy fibrant" diagrams on a FOLDS-signature coincide with structures for the corresponding axiomatic signature. But for space reasons, we postpone this to Appendix C. Instead we sketch the results of the translation for the examples in Fig. 1.

Examples 4.4. All three examples have only one sort of rank 0, so that $\mathcal{L}_\perp = \{O\} = \mathbf{1}$, and $M_\perp$ consists of a single type MO. Moreover, since all three examples have only one sort A of rank 1 that depends on O twice, their derivatives $\mathcal{L}'_{MO}$ have $(\mathcal{L}'_{MO})_\perp = MO \times MO$, and $(M')_\perp$ is a single type family $MA : MO \times MO \to \mathcal{U}$. Finally, since all three have height 3, $(\mathcal{L}'_{MO})'_{MA}$ has height 1, hence is just a single type.

- For $\mathcal{L}_{\mathrm{rg}}$, this type is $\sum_{(x:MO)} MA(x,x)$, so that a structure is completed by a type family
$$MI : \left(\textstyle\sum_{(x:MO)} MA(x,x)\right) \to \mathcal{U}.$$
- For $\mathcal{L}_{\mathrm{cat}}$, this type is
$$\left(\textstyle\sum_{(x,y,z:MO)} MA(x,y) \times MA(y,z) \times MA(x,z)\right) + \left(\textstyle\sum_{(x:MO)} MA(x,x)\right),$$
so that a structure is completed by a type family MI as above together with
$$MT : \left(\textstyle\sum_{(x,y,z:MO)} MA(x,y) \times MA(y,z) \times MA(x,z)\right) \to \mathcal{U}.$$
- Finally, for $\mathcal{L}_{\mathrm{cat+E}}$, this type is
$$\left(\textstyle\sum_{(x,y,z:MO)} MA(x,y) \times MA(y,z) \times MA(x,z)\right) + \left(\textstyle\sum_{(x:MO)} MA(x,x)\right) + \left(\textstyle\sum_{(x,y:MO)} MA(x,y) \times MA(x,y)\right),$$
so that a structure is completed by type families MI and MT as above together with
$$ME : \left(\textstyle\sum_{(x,y:MO)} MA(x,y) \times MA(x,y)\right) \to \mathcal{U}.$$

5 (Iso)morphisms of structures

The definition of structures for signatures doesn't require the fact that signatures form an s-category. But defining *morphisms* of structures will require the pullback of an $\mathcal{M}$-structure along a morphism $\alpha : \mathcal{L} \to \mathcal{M}$ of signatures, defined as follows.

Definition 5.1. For any $\alpha : \hom_{\mathsf{Sig}(n)}(\mathcal{L}, \mathcal{M})$, we define the **pullback** $\alpha^* : \mathsf{Str}(\mathcal{M}) \to \mathsf{Str}(\mathcal{L})$ inductively as follows.

If $n :\equiv 0$, then let $\alpha^* : \mathsf{Str}(\mathcal{M}) \to \mathsf{Str}(\mathcal{L})$ be the identity.

If $n > 0$, consider $M : \mathsf{Str}(\mathcal{M})$. We let $(\alpha^* M)_\perp$ be $M_\perp \circ \alpha_\perp$. By induction, the morphism

$$\alpha'_{M_\perp} : \hom_{\mathsf{Sig}(n-1)}(\mathcal{L}'_{M_\perp \circ \alpha_\perp}, \mathcal{M}'_{M_\perp})$$

produces a $(\alpha'_{M_\perp})^* : \mathsf{Str}(\mathcal{M}'_{M_\perp}) \to \mathsf{Str}(\mathcal{L}'_{M_\perp \circ \alpha_\perp})$, so we set $(\alpha^* M)' :\equiv (\alpha'_{M_\perp})^* M'$.

Pullback is functorial: pullback along a composition of signature morphisms is the composition of pullbacks, and pullback along an identity morphism is the identity.

We now inductively define *morphisms* between structures of a given signature, making $\mathsf{Str}(\mathcal{L})$ into an s-category.

Definition 5.2 (Morphism of structures). Consider $\mathcal{L} : \mathsf{Sig}(n)$ and $M, N : \mathsf{Str}(\mathcal{L})$.

When $n :\equiv 0$, we let $\hom_{\mathsf{Str}(\mathcal{L})}(M, N) :\equiv \mathbf{1}$.

When $n > 0$, a morphism $f : \hom_{\mathsf{Str}(\mathcal{L})}(M, N)$ consists of

1. $f_\perp : \prod_{(K:\mathcal{L}_\perp)} M_\perp(K) \to N_\perp(K)$
2. $f' : \hom_{\mathsf{Str}(\mathcal{L}'_{M_\perp})}(M', (\mathcal{L}'_{f_\perp})^* N')$.

Lemma 5.3. *For a signature $\mathcal{L}$ and $\mathcal{L}$-structures M and N, the type of morphisms from M to N is fibrant.* □

As a stepping-stone to our SIP for univalent $\mathcal{L}$-structures, we show that *all* $\mathcal{L}$-structures satisfy a tautological "levelwise" form of univalence.

Definition 5.4 (Isomorphism of structures). Consider $\mathcal{L} : \mathrm{Sig}(n)$ and $M, N : \mathrm{Str}(\mathcal{L})$.

If $n :\equiv 0$, we define every $f : \mathrm{hom}_{\mathsf{Str}(\mathcal{L})}(M, N)$ to be an $\mathcal{L}$-isomorphism. That is, we define $\mathsf{isIso}_{\mathcal{L}}(f) :\equiv \mathbf{1}$.

For $n > 0$, $f : \mathrm{hom}_{\mathsf{Str}(\mathcal{L})}(M, N)$ is an *$\mathcal{L}$-isomorphism* when

1. $f_\perp(K)$ is an equivalence of types for all $K : \mathcal{L}_\perp$, and
2. f' is an $\mathcal{L}'_{M_\perp}$-isomorphism.

That is, let

$$\mathsf{isIso}_{\mathcal{L}}(f) :\equiv \left(\prod_{(K:L_\perp)} \mathsf{isEquiv}(f_\perp(K)) \right) \times \mathsf{isIso}_{\mathcal{L}'_{M_\perp}}(f').$$

We denote the type of $\mathcal{L}$-isomorphisms between two $\mathcal{L}$-structures M, N by $M \cong_{\mathcal{L}} N$, or simply $M \cong N$.

Lemma 5.5. *For any morphism $f : M \to N$ between two $\mathcal{L}$-structures, the type $\mathsf{isIso}_{\mathcal{L}}(f)$ is fibrant and a proposition.* □

Definition 5.6 (Identity isomorphisms). When $n :\equiv 0$, we define i_M to be the canonical element in $\mathbf{1}$.

Otherwise, we have a strict equality $1_{\mathcal{L}'_{M_\perp}} \overset{\mathrm{s}}{=} \mathcal{L}'_{1_{M_\perp}}$ (from functoriality of $\mathcal{L}'$), hence $u : (1_{\mathcal{L}'_{M_\perp}})^* M' \overset{\mathrm{s}}{=} (\mathcal{L}'_{1_{M_\perp}})^* M'$. We also have a strict equality $v : M' \overset{\mathrm{s}}{=} (1_{\mathcal{L}'_{M_\perp}})^* M'$ (by the functoriality of pullback). Then we set i_M to be the pair $(1_{M_\perp}, \mathsf{idtoiso}(v \cdot u))$.

Now define $\mathsf{idtoiso} : \prod_{(M,N:\mathsf{Str}(\mathcal{L}))} (M = N) \to (M \cong N)$ by sending refl_M to i_M.

Proposition 5.7. *For structures M, N of a signature $\mathcal{L}$, the canonical map*

$$\mathsf{idtoiso}_{M,N} : (M = N) \to (M \cong N)$$

is an equivalence of types.

Proof. When $n :\equiv 0$, idtoiso is a function $\mathbf{1} \to \mathbf{1}$ and so an equivalence.

Let $\mathsf{ua} : (M_\perp \cong N_\perp) \to (M_\perp = N_\perp)$ denote the function given by the univalence axiom. First, we show that for any $e : M_\perp \cong N_\perp$, we have $\mathsf{ua}(e)^{-1}{}_*(N') = (\mathcal{L}'_e)^* N'$ where $\mathsf{ua}(e)^{-1}{}_*(N')$ denotes the transport of N' along the identification $\mathsf{ua}(e)^{-1}$. Consider the diagram of functions given in Figure 2. The square in the diagram commutes (up to $=$) since both functions $(M_\perp = N_\perp) \to \mathsf{Str}(\mathcal{L}'_{N_\perp}) \to \mathsf{Str}(\mathcal{L}'_{M_\perp})$ send $\mathsf{refl}_{M_\perp}$ to $1_{\mathsf{Str}(\mathcal{L}'_M)}$ (by strict functoriality of the pullback). Precomposing these with ua, we find that $(\mathcal{L}'_e)^* N' = \mathsf{ua}(e)^{-1}{}_*(N')$. Now we have that

$$\begin{aligned}
(M = N) &= \sum_{p : M_\perp = N_\perp} M' = p^{-1}{}_*(N') \\
&= \sum_{e : M_\perp \cong N_\perp} M' = \mathsf{ua}(e)^{-1}{}_*(N') \\
&= \sum_{e : M_\perp \cong N_\perp} M' = (\mathcal{L}'_e)^* N' \\
&= \sum_{e : M_\perp \cong N_\perp} M' \cong (\mathcal{L}'_e)^* N' \\
&\equiv (M \cong N)
\end{aligned}$$

where the second identification is the univalence axiom and the fourth is our inductive hypothesis. This equivalence, from left to right, is $\mathsf{idtoiso}_{M,N}$. □

Proposition 5.7 relies on the univalence axiom; conversely, the univalence axiom can be recovered as an instance of Proposition 5.7, for the signature consisting of just one sort.

Example 5.8. When precategories are regarded as $\mathcal{L}_{\mathrm{cat}}$-structures, their isomorphisms are the *isomorphisms of precategories* from [2, Def. 6.9] and [11, Def. 9.4.8]: functors that induce equivalences on hom-types and also equivalences on types of objects (relative to homotopical *identifications* of objects, not isomorphisms in the category structure).

The analogue for $\mathcal{L}$-structures of *equivalences* of precategories, called *very split-surjective morphisms* of $\mathcal{L}$-structures, will be introduced in Section 7. Our main result, Theorem 7.8, will be that between univalent $\mathcal{L}$-structures these are also equivalent to identifications. However, first we have to define univalence of $\mathcal{L}$-structures.

6 Isomorphism and Univalence

In this section we define isomorphism of objects within an $\mathcal{L}$-structure. We then define a structure to be univalent when isomorphism coincides with identification of objects.

Let M be an $\mathcal{L}$-structure, $K : \mathcal{L}_\perp$, and $a, b : M_\perp K$. To define isomorphisms from a to b, we consider a new $\mathcal{L}$-structure obtained by adding to M one element at sort K: a "joker" element. We can substitute this new element by a or by b; below, we call the obtained structures $\partial_a M$ and $\partial_b M$, respectively. An isomorphism from a to b will be defined below to be an isomorphism of structures from $\partial_a M$ to $\partial_b M$ that is the identity on all the sorts not depending on the joker element. Intuitively, this means that a and b are isomorphic when they are interchangeable in any "grammatical expression" permitted by our "vocabulary" $\mathcal{L}$. To make this intuition formal, we need two auxiliary definitions:

Definition 6.1. Consider $L : \mathcal{U}$, $K : L$, $M : L \to \mathcal{U}$, $a : M(K)$. We define the **indicator function of** K to be

$$[K] :\equiv \lambda x.\, (x = K) : L \to \mathcal{U}$$

and we define the function $a : \prod_{(x:L)} [K](x) \to M(x)$ by sending $\mathsf{refl}_K : [K](K)$ to $a : M(K)$.

Below we consider the pointwise disjoint union $M + [K]$ in $L \to \mathcal{U}$, the canonical injection $\iota_M : \prod_{(x:L)} M(x) \to (M + [K])(x)$, and the induced function $\langle 1_M, a \rangle : \prod_{(x:L)} (M + [K])(x) \to M(x)$.

Definition 6.2. Consider $\mathcal{L} : \mathrm{Sig}(n{+}1)$, $K : \mathcal{L}_\perp$, $M : \mathrm{Str}(\mathcal{L})$, $a : M_\perp(K)$. Define

$$\partial_a M :\equiv (\mathcal{L}'_{\langle 1_{M_\perp}, a \rangle})^* M' : \mathrm{Str}(\mathcal{L}'_{M_\perp + [K]}).$$

$$
\begin{tikzcd}
(M_\perp \cong N_\perp) \arrow[r, "\mathsf{ua}"] & (M_\perp = N_\perp) \arrow[r, "\mathsf{idtoiso}"] \arrow[d, "(-)^{-1}"] & (M_\perp \cong N_\perp) \arrow[r, "\mathcal{L}'_-"] & \hom_{\mathsf{Sig}(n)}(\mathcal{L}'_{M_\perp}, \mathcal{L}'_{N_\perp}) \arrow[d, "(-)^*"] \\
& (M_\perp = N_\perp) \arrow[rr, "(-)_*"] & & \mathsf{Str}(\mathcal{L}'_{N_\perp}) \to \mathsf{Str}(\mathcal{L}'_{M_\perp})
\end{tikzcd}
$$

Figure 2. Diagram for Proof of Proposition 5.7

Note that we require the type $\mathcal{L}(0)$ to be fibrant so that the fibrant indicator function $[K]$ exists.

Now we can define the type of isomorphisms between objects within an $\mathcal{L}$-structure:

Definition 6.3 (Isomorphism within a structure). Consider $\mathcal{L} : \mathsf{Sig}(n+1)$, $K : \mathcal{L}_\perp$, $M : \mathsf{Str}(\mathcal{L})$, $a, b : M_\perp(K)$. We define the type

$$a \cong^M_K b :\equiv \sum_{p:\, \partial_a M = \partial_b M} \epsilon_a^{-1} \cdot (\mathcal{L}'_{\iota_{M_\perp}})^* p \cdot \epsilon_b =_{M'=M'} \mathsf{refl}_{M'},$$

where ϵ_x is the concatenated identification

$$
\begin{aligned}
(\mathcal{L}'_{\iota_{M_\perp}})^* \partial_x M' &\equiv (\mathcal{L}'_{\iota_{M_\perp}})^* (\mathcal{L}'_{\langle 1_{M_\perp}, x\rangle})^* M' \\
&= (\mathcal{L}'_{\langle 1_{M_\perp}, x\rangle \circ \iota_{M_\perp}})^* M' = (\mathcal{L}'_{1_{M_\perp}})^* M' = M'.
\end{aligned}
$$

Lemma 6.4. *The type $a \cong^M_K b$ of Definition 6.3 is fibrant.* □

Remark 6.5. Using identification instead of isomorphism of structures in Definition 6.3 is justified by Proposition 5.7.

We now define *univalence of $\mathcal{L}$-structures*. For this, we first need to define the canonical map from identifications to isomorphisms between objects within an $\mathcal{L}$-structure.

Definition 6.6 (Identity isomorphism). For $\mathcal{L} : \mathsf{Sig}(n+1)$, $K : \mathcal{L}_\perp$, $M : \mathsf{Str}(\mathcal{L})$, and $m : M_\perp(K)$, we define the isomorphism $1 : m \cong^M_K m$. Let $M : \mathsf{Str}(\mathcal{L})$. For any $a : M_\perp(K)$, we have $\mathsf{refl}_{\partial_a M} : \partial_a M = \partial_a M$. Then

$$
\begin{aligned}
&\epsilon_a^{-1} \cdot (\mathcal{L}'_{\iota_M})^* (\mathsf{refl}_{\partial_a M}) \cdot \epsilon_a \\
&\quad \overset{\mathsf{s}}{=} \epsilon_a^{-1} \cdot \mathsf{refl}_{(\mathcal{L}'_{\iota_M})^* \partial_a M} \cdot \epsilon_a \\
&\quad = \mathsf{refl}_{M'} \ ,
\end{aligned}
$$

where the second identification uses the groupoidal properties of types. This gives the desired isomorphism.

Definition 6.7. Consider $\mathcal{L} : \mathsf{Sig}(n+1)$, $K : \mathcal{L}_\perp$, $M : \mathsf{Str}(\mathcal{L})$. For any $a, b : M_\perp(K)$, let $\mathsf{idtoiso}_{a,b} : (a =_{M_\perp(K)} b) \to (a \cong^M_K b)$ be the function which sends refl_a to the identity isomorphism exhibited in Definition 6.6.

We say that M is **univalent at** K if for all $a, b : M_\perp(K)$, $\mathsf{idtoiso}_{a,b} : (a =_{M_\perp(K)} b) \to (a \cong^M_K b)$ is an equivalence.

Definition 6.8 (Univalence of structures). We define by induction what it means for a structure of a signature $\mathcal{L} : \mathsf{Sig}(n)$ to be univalent.

When $n :\equiv 0$, every structure $M : \mathsf{Str}(\mathcal{L})$ is univalent.

Otherwise, a structure $M : \mathsf{Str}(\mathcal{L})$ is univalent if M is univalent at all $K : \mathcal{L}_\perp$ and M' is univalent.

Let $\mathsf{uStr}(\mathcal{L})$ denote the type of univalent structures of $\mathcal{L}$.

Lemma 6.9. *Let $\mathcal{L}$ be a signature.*

- *The type $\mathsf{uStr}(\mathcal{L})$ is fibrant.*
- *For any $\mathcal{L}$-structure, "being univalent" is a proposition.*
- *Identification of univalent $\mathcal{L}$-structures corresponds to identification of the underlying $\mathcal{L}$-structures.* □

Example 6.10. Suppose $\mathcal{L}$ has height 1, hence is just a type $\mathcal{L}_\perp$. Consider an $\mathcal{L}$-structure $M : \mathcal{L}_\perp \to \mathcal{U}$ and $a, b : M(K)$. Then $\partial_a M$ and $\partial_b M$ are structures for the trivial signature of height 0, hence uniquely identified; thus $(a \cong^M_K b) = \mathbf{1}$. So any structure of a signature $\mathcal{L}$ of height 1 is univalent just when it consists entirely of propositions.

Example 6.11. Recall from Examples 4.4 that for $\mathcal{L} = \mathcal{L}_{\mathrm{cat+E}}$, we have $\mathcal{L}_\perp = \mathbf{1}$, $M_\perp = MO : \mathcal{U}$, $\mathcal{L}'_{MO\,\perp} = MO \times MO$, and $(M')_\perp = MA : MO \times MO \to \mathcal{U}$, while and M'' consists of the sorts $T_{x,y,z}(f,g,h)$, $I_x(f)$, and $E_{x,y}(f,g)$. By Example 6.10, M'' is univalent just when all these types are propositions. Now for any $a, b : MO$, we have

$$(MA + [A(a,b)])(x,y) = MA(x,y) + ((a = x) \times (b = y)).$$

Thus, the height-1 signature $(\mathcal{L}'_{MO})'_{MA+[A(a,b)]}$ is

$$
\begin{aligned}
&\big(\textstyle\sum_{(x,y,z:MO)} (MA(x,y) + ((a = x) \times (b = y))) \\
&\qquad \times (MA(y,z) + ((a = y) \times (b = z))) \\
&\qquad \times (MA(x,z) + ((a = x) \times (b = z)))\big) \\
&\quad + \big(\textstyle\sum_{(x:MO)} (MA(x,x) + ((a = x) \times (b = x)))\big) \\
&\quad + \big(\textstyle\sum_{(x,y:MO)} (MA(x,y) + ((a = x) \times (b = y))) \\
&\qquad \times (MA(x,y) + ((a = x) \times (b = y)))\big).
\end{aligned}
$$

By distributing $\sum$ and $\times$ over $+$ and contracting some singletons, this is equivalent to

$$\big(\textstyle\sum_{(x,y,z:MO)} MA(x,y) \times MA(y,z) \times MA(x,z)\big) \tag{22}$$
$$+ \big(\textstyle\sum_{(z:MO)} MA(b,z) \times MA(a,z)\big) \tag{23}$$
$$+ \big(\textstyle\sum_{(x:MO)} MA(x,a) \times MA(x,b)\big) \tag{24}$$
$$+ \big(\textstyle\sum_{(y:MO)} MA(a,y) \times MA(y,b)\big) \tag{25}$$
$$+ \big((a = b) \times MA(a,b)\big) \tag{26}$$
$$+ \big(MA(a,a) \times (b = b)\big) \tag{27}$$
$$+ \big((a = a) \times MA(b,b)\big) \tag{28}$$
$$+ \big((a = b) \times (a = a) \times (b = b)\big) \tag{29}$$
$$+ \big(\textstyle\sum_{(x:MO)} MA(x,x)\big) \tag{30}$$

$$+\big((a=b)\big) \tag{31}$$
$$+\big(\textstyle\sum_{(x,y:MO)} MA(x,y)\times MA(x,y)\big) \tag{32}$$
$$+\big(MA(a,b)\big) \tag{33}$$
$$+\big(MA(a,b)\big) \tag{34}$$
$$+\big((a=a)\times(b=b)\big) \tag{35}$$

Thus for $f, g : MA(a,b)$, an identification $\partial_f M = \partial_g M$ consists of equivalences between instances of the predicates MT, MI, ME indexed over the types (22)–(35). The condition on restriction along ι says that the equivalences corresponding to (22), (30), and (32) are the identity, while those corresponding to (23)–(25), (26)–(29), (31), and (33)–(35) yield respectively the equivalences (11)–(13), (14)–(17), (18), and (19)–(21) from Section 2.3. Hence, isomorphisms $f \cong g$ in the sense of Definition 6.3 coincide with the FOLDS-isomorphisms from Definition 2.11.

Now moving back down to the bottom rank, an $(\mathcal{L}'_{MO})$-structure consists of $MA : MO \times MO \to \mathcal{U}$ together with appropriately typed families MT, MI, and ME. Since $(MO + [O]) = MO + \mathbf{1}$, for $a : MO$ the 0^{th} rank of $\partial_a M$ is

$$(\partial_a M)A : (MO+\mathbf{1})\times(MO+\mathbf{1}) \to \mathcal{U}$$

or equivalently

$$(\partial_a M)A : (MO\times MO) + MO + MO + \mathbf{1} \to \mathcal{U}$$

consisting of the types $\{MA(x,y)\}_{x,y:MO}$, $\{MA(a,y)\}_{y:MO}$, $\{MA(x,a)\}_{x:MO}$, and $MA(a,a)$. The 1^{st} rank consists of MT, MI, and ME pulled back appropriately to these families. Thus, an identification $\partial_a M = \partial_b M$ consists of equivalences

$$MA(x,y) \simeq MA(x,y) \tag{36}$$
$$MA(x,a) \simeq MA(x,b) \tag{37}$$
$$MA(a,y) \simeq MA(b,y) \tag{38}$$
$$MA(a,a) \simeq MA(b,b) \tag{39}$$

for all $x, y : MO$ that respect the predicates MT, MI, ME. The condition on restriction along ι says that the equivalences (36) are the identity, while the remaining (37)–(39) correspond respectively to the equivalences $\phi_{x\bullet}$, $\phi_{\bullet y}$, and $\phi_{\bullet\bullet}$ from Section 2.3. Finally, respect for MT, MI, ME specializes to (3)–(10) together with analogous equivalences for E that are trivial under "standardness" of identifications. Thus, isomorphisms $a \cong b$ in the sense of Definition 6.3 coincide with the FOLDS-isomorphisms from Definition 2.6.

7 Results

Our first two results give truncation bounds for types within, and of, univalent structures:

Theorem 7.1. *Let $\mathcal{L} : \mathsf{Sig}(n+1)$, $M : \mathsf{uStr}(\mathcal{L})$, $K : \mathcal{L}_\perp$. Then $M_\perp(K)$ is an $(n-1)$-type.*

Theorem 7.2. *Let $\mathcal{L} : \mathsf{Sig}(n)$. The type of univalent $\mathcal{L}$-structures is an $(n-1)$-type.*

Proof of Theorems 7.1 and 7.2. Define the following types.

$$P(n) :\equiv \prod_{\mathcal{L}:\mathsf{Sig}(n+1)} \ \prod_{M:\mathsf{uStr}(\mathcal{L})} \ \prod_{K:\mathcal{L}_\perp} \text{is-}(n-1)\text{-type}(M_\perp(K))$$
$$Q(n) :\equiv \prod_{\substack{\mathcal{M},\mathcal{N}:\mathsf{Sig}(n)\\ \alpha:\hom(\mathcal{M},\mathcal{N})}} \ \prod_{N:\mathsf{uStr}(\mathcal{N})} \text{is-}(n-2)\text{-type}(\alpha^*N = \alpha^*N)$$

The type $P(n)$ is the statement of Theorem 7.1, and the type $Q(n)$ implies the statement of Theorem 7.2 by [11, Thm. 7.2.7]. We prove $P(n)$ and $Q(n)$ simultaneously.

For $P(n)$, we need to show that $a =_{M_\perp K} b$ is an $(n-2)$-type for all $\mathcal{L} : \mathsf{Sig}(n+1), M : \mathsf{uStr}(\mathcal{L}), K : \mathcal{L}_\perp, a, b : M_\perp K$. But since M is univalent, this type is equivalent to

$$(a \cong^M_K b) \equiv \sum_{e:\partial_a M = \partial_b M} \epsilon_a^{-1}\cdot(\mathcal{L}'_{\iota_M})^* p \cdot \epsilon_b =_{M'=M'} \mathsf{refl}_{M'}.$$

Thus, it will suffice to show that $\partial_a M = \partial_b M$ and $\epsilon_a^{-1}\cdot(\mathcal{L}'_{\iota_M})^* p\cdot\epsilon_b =_{M'=M'} \mathsf{refl}_{M'}$ are $(n-2)$-types.

To show $P(0)$ and $Q(0)$ consider $\mathcal{L} : \mathsf{Sig}(1), M : \mathsf{uStr}(\mathcal{L}), K : \mathcal{L}_\perp, a, b : M_\perp K, \mathcal{M}, \mathcal{N} : \mathsf{Sig}(0), \alpha : \hom(\mathcal{M},\mathcal{N}), N : \mathsf{uStr}(\mathcal{N})$. We have that $M', \partial_a M, \partial_b M, \alpha^*N : \mathbf{1}$ so the types $\partial_a M = \partial_b M$, $\epsilon_a^{-1}\cdot(\mathcal{L}'_{\iota_M})^* p\cdot\epsilon_b =_{M'=M'} \mathsf{refl}_{M'}$, and $\alpha^*N = \alpha^*N$ are contractible. Thus, $P(0)$ and $Q(0)$ hold.

Suppose that $P(n)$ and $Q(n)$ hold. We first show $Q(n+1)$. Consider $\mathcal{M}, \mathcal{N} : \mathsf{Sig}(n+1), \alpha : \hom(\mathcal{M},\mathcal{N}), N : \mathsf{uStr}(\mathcal{N})$. We have that

$$\begin{aligned}(\alpha^*N = \alpha^*N) &\simeq \textstyle\sum_{e:(\alpha^*N)_\perp = (\alpha^*N)_\perp} (\alpha^*N)' = e_*(\alpha^*N)' \\ &\equiv \textstyle\sum_{e:(N_\perp\circ\alpha_\perp)=(N_\perp\circ\alpha_\perp)} (\alpha'_{N_\perp})^*N' = e_*(\alpha'_{N_\perp})^*N'.\end{aligned}$$

Our inductive hypothesis $Q(n)$ ensures that $(\alpha'_{N_\perp})^*N' = (\alpha'_{N_\perp})^*N'$ is an $(n-2)$-type, and hence $(\alpha'_{N_\perp})^*N' = e_*(\alpha'_{N_\perp})^*N'$ is an $(n-1)$-type by [11, Thm. 7.2.7]. It remains to show that $(N_\perp\circ\alpha_\perp) = (N_\perp\circ\alpha_\perp)$ is an $(n-1)$-type. Note that N is a univalent structure of an $(n+1)$-signature, and our inductive hypothesis $P(n)$ then implies that for all $K : \mathcal{N}_\perp$, the type $N_\perp(K)$ is an $(n-1)$-type. Then since $(N_\perp\circ\alpha_\perp)$ is a function which takes values in $(n-1)$-types, we can conclude that $(N_\perp\circ\alpha_\perp) = (N_\perp\circ\alpha_\perp)$ is an $(n-1)$-type [11, Thm. 7.1.9]. Thus, $Q(n+1)$ holds.

To show that $P(n+1)$ holds, consider $\mathcal{L} : \mathsf{Sig}(n+2), M : \mathsf{uStr}(\mathcal{L}), K : \mathcal{L}_\perp, a, b : M_\perp K$. By [11, Thm. 7.2.7], $Q(n+1)$ implies that $\partial_a M = \partial_b M$ and $\epsilon_a^{-1}\cdot(\mathcal{L}'_{\iota_M})^* p\cdot\epsilon_b =_{M'=M'} \mathsf{refl}_{M'}$ are $(n-2)$-types. Therefore, $P(n+1)$ holds. □

Example 7.3. For the signature $\mathcal{L}_{\text{cat+E}}$ of height 3, Theorem 7.1 states that the type of objects of a univalent $\mathcal{L}_{\text{cat+E}}$-structure, and hence also of a univalent FOLDS-category, is a 1-type. Theorem 7.2 states that the type of univalent $\mathcal{L}_{\text{cat+E}}$-structures, and hence also the type of univalent FOLDS-categories (as a subtype of the former), is a 2-type.

We now move on to our Higher Structure Identity Principle (Theorem 7.8), which requires a notion of equivalence *between* structures.

We start with definitions.

Definition 7.4. Consider $\mathcal{L} : \mathsf{Sig}(n)$, $S, T : \mathsf{Str}(L)$. If $n :\equiv 0$, the type of **sections** of a morphism $\mu : \hom_{\mathcal{L}}(S,T)$ is $\mathbf{1}$. For $n > 0$, a **section** of a morphism $\mu : \hom_{\mathcal{L}}(S,T)$ consists of a section of the function $\mu_\perp(K)$ for every $K : \mathcal{L}_\perp$ together with a section of μ'. That is, let

$$\mathsf{Section}(\mu) :\equiv \left(\prod_{K:\mathcal{L}_\perp} \mathsf{Section}(\mu_\perp K)\right) \times \mathsf{Section}(\mu').$$

Say that a morphism $\mu : S \to T$ is **very split-surjective** when it has a section, and let $S \twoheadrightarrow T$ denote the type of **very split-surjective morphisms from** S **to** T, that is, of pairs of a morphism $\mu : S \to T$ together with a section of μ.

Lemma 7.5. *Consider $\mathcal{L} : \mathsf{Sig}(n)$, $S, T : \mathsf{Str}(L)$. The type $S \twoheadrightarrow T$ of very split surjective morphisms is fibrant.* □

Definition 7.6 (From identifications to very split-surjective morphisms). Consider $\mathcal{L} : \mathsf{Sig}(n)$, $M, N : \mathsf{Str}(\mathcal{L})$. We construct a morphism $\mathsf{idtovsm} : (M = N) \to (M \twoheadrightarrow N)$.

We construct an identity morphism $1_M : \hom_{\mathsf{Str}(\mathcal{L})}(M, M)$ by induction. When $n :\equiv 0$, we let 1_M be the canonical element in $\hom_{\mathcal{L}}(M, M)$. When $n > 0$, let $1_M :\equiv (\lambda K.1_{M_\perp(K)}, 1_{M'})$. This identity morphism always has a section. We define $\mathsf{idtovsm} : (M = N) \to (M \twoheadrightarrow N)$ by sending refl_M to 1_M.

Lemma 7.7. *Let $\mathcal{L} : \mathsf{Sig}(n+1)$, $M, N : \mathsf{Str}(\mathcal{L})$, $f_\perp : M_\perp \to N_\perp$, and $e : M' = (\mathcal{L}'_{f_\perp})^* N'$. Then for $x, y : M_\perp(K)$, an isomorphism $f_\perp x \cong^N_K f_\perp y$ produces an isomorphism $x \cong^M_K y$.*

Proof. We use path induction on the identification $e : M' = (\mathcal{L}'_{f_\perp})^* N'$ to assume that $M' \equiv (\mathcal{L}'_{f_\perp})^* N'$.

Consider the following diagram whose cells commute up to $\overset{s}{=}$ or $=$, as pictured.

$$\begin{tikzcd}
M_\perp \arrow[r, "\iota_{M_\perp}"] \arrow[d, "f_\perp"] \arrow[rr, bend left, "1_{M_\perp}"] \arrow[dr, phantom, "\overset{s}{=}"] & M_\perp + [K] \arrow[r, "\langle 1, x\rangle"] \arrow[d, "f_\perp + 1"] \arrow[dr, phantom, "="] & M_\perp \arrow[d, "f_\perp"] \\
N_\perp \arrow[r, "\iota_{N_\perp}"] \arrow[rr, bend right, "1_{N_\perp}"'] & N_\perp + [K] \arrow[r, "\langle 1, f_\perp(K) x\rangle"] & N_\perp
\end{tikzcd} \tag{40}$$

This diagram commutes 2-dimensionally, which is to say that the "pasting" of all four displayed identities is strictly equal to the strict (indeed, judgemental) equality $f_\perp \circ 1_{M_\perp} \overset{s}{=} 1_{N_\perp} \circ f_\perp$. Applying the composite s-functor $\mathsf{Str}(\mathcal{L}'_{\_})$, we obtain:

$$\begin{tikzcd}
\mathsf{Str}(\mathcal{L}'_{M_\perp}) \arrow[dr, phantom, "\overset{s}{=}(\alpha)"] & \mathsf{Str}(\mathcal{L}'_{M_\perp + [K]}) \arrow[l, "(\mathcal{L}'_{\iota_{M_\perp}})^*"'] \arrow[dr, phantom, "=(\beta_x)"] & \mathsf{Str}(\mathcal{L}'_{M_\perp}) \arrow[l, "(\mathcal{L}'_{\langle 1, x\rangle})^*"'] \arrow[ll, bend right, "1_{\mathsf{Str}(\mathcal{L}'_{M_\perp})}"', "\overset{s}{=}(\epsilon_x)"] \\
\mathsf{Str}(\mathcal{L}'_{N_\perp}) \arrow[u, "(\mathcal{L}'_{f_\perp})^*"] & \mathsf{Str}(\mathcal{L}'_{N_\perp + [K]}) \arrow[l, "(\mathcal{L}'_{\iota_{N_\perp}})^*"] \arrow[u, "(\mathcal{L}'_{f_\perp + 1})^*"] & \mathsf{Str}(\mathcal{L}'_{N_\perp}) \arrow[l, "(\mathcal{L}'_{\langle 1, f_\perp x\rangle})^*"] \arrow[u, "(\mathcal{L}'_{f_\perp})^*"] \arrow[ll, bend left, "1_{\mathsf{Str}(\mathcal{L}'_{N_\perp})}", "\overset{s}{=}(\epsilon_{f_\perp x})"']
\end{tikzcd} \tag{41}$$

which commutes in the same way. Moreover, the upper and lower strict equalities in this diagram are ϵ_x and $\epsilon_{f_\perp x}$ respectively; we call the others α and β_x.

We have an analogous diagram for y, in which the left-hand square α is the same.

Then since $\partial_{f_\perp(K)x} N \equiv (\mathcal{L}'_{\langle 1, f_\perp(K)x\rangle})^* N'$, $M' \equiv (\mathcal{L}'_{f_\perp})^* N'$, and $\partial_x M \equiv (\mathcal{L}'_{\langle 1,x\rangle})^* M'$, we have an identification

$$\beta_x N : (\mathcal{L}'_{f_\perp + 1})^* \partial_{f_\perp(K)x} N = \partial_x M.$$

The same can be shown for y.

Consider an isomorphism $f_\perp x \cong^N_K f_\perp y$ which consists of (1) an identification $i : \partial_{f_\perp x} N = \partial_{f_\perp y} N$ and (2) a identification j between $(\mathcal{L}'_{\iota_{N_\perp}})^* i$ and the concatenation

$$(\mathcal{L}'_{\iota_{N_\perp}})^* (\mathcal{L}'_{\langle 1, f_\perp x\rangle})^* N' \overset{\epsilon_{f_\perp x}}{\overset{s}{=}} N' \overset{\epsilon_{f_\perp y}^{-1}}{\overset{s}{=}} (\mathcal{L}'_{\iota_{N_\perp}})^* (\mathcal{L}'_{\langle 1, f_\perp y\rangle})^* N'$$

(which is strict, though i is not).

We need to construct an isomorphism $x \cong^M_K y$ which consists of (1) an identification $k : \partial_x M = \partial_y M$ and (2) an identification $(\mathcal{L}'_{\iota_{M_\perp}})^* k = \epsilon_x \cdot \epsilon_y^{-1}$.

The first component, k, of our desired isomorphism $x \cong y$ is the following concatenation:

$$\begin{aligned}
(\mathcal{L}'_{\langle 1,x\rangle})^* (\mathcal{L}'_{f_\perp})^* N' &\overset{\beta_x}{=} (\mathcal{L}'_{f_\perp + 1})^* (\mathcal{L}'_{\langle 1, f_\perp x\rangle})^* N' \\
&\overset{(\mathcal{L}'_{f_\perp + 1})^* i}{=} (\mathcal{L}'_{f_\perp + 1})^* (\mathcal{L}'_{\langle 1, f_\perp y\rangle})^* N' \\
&\overset{\beta_y^{-1}}{=} (\mathcal{L}'_{\langle 1,y\rangle})^* (\mathcal{L}'_{f_\perp})^* N'
\end{aligned}$$

Now we need an identification $(\mathcal{L}'_{\iota_{M_\perp}})^* k = \epsilon_x \cdot \epsilon_y^{-1}$. Consider the commutative diagram in Figure 3 where straight lines denote strict equalities, squiggly lines denote identifications, and double (squiggly) lines denote identifications between identifications. The 2-dimensional identification labelled ν arises from naturality of identifications. The 2-dimensional identifications labeled σ arise from the 2-dimensional commutativity of Diag. (41). The concatenation of the three top horizontal identifications in the diagram is $(\mathcal{L}'_{\iota_{M_\perp}})^* k$. The diagram exhibits an identification of this with $\epsilon_x \cdot \epsilon_y^{-1}$.

Thus, we find that $(\mathcal{L}'_{\iota_{M_\perp}})^* k = \epsilon_x \cdot \epsilon_y^{-1}$, and we have constructed an isomorphism $x \cong^M_K y$. □

Theorem 7.8 (Higher SIP). *Consider $\mathcal{L} : \mathsf{Sig}(n)$ and $M, N : \mathsf{Str}(\mathcal{L})$ such that M is univalent. The morphism $\mathsf{idtovsm} : (M = N) \to (M \twoheadrightarrow N)$ is an equivalence.*

Proof. First, we construct the evident forgetful function $U_f : \mathsf{isIso}(f) \to \mathsf{Section}(f)$ by induction for each $n : \mathbb{N}^s$, $\mathcal{L} : \mathsf{Sig}(n)$, $M : \mathsf{uStr}(\mathcal{L})$, $N : \mathsf{Str}(\mathcal{L})$, and $f : \hom_{\mathsf{Str}(\mathcal{L})}(M, N)$. At $n :\equiv 0$, let $U_f : \mathsf{isIso}(f) \to \mathsf{Section}(f)$ be the identity function on $\mathbf{1}$. Otherwise, we have the following

$$\mathsf{isIso}(f) :\equiv \prod_{K:\mathcal{L}_\perp} \mathsf{isEquiv}(f_\perp(K)) \times \mathsf{isIso}(f'),$$

$$
\begin{tikzcd}
(\mathcal{L}'_{\iota_{M_\perp}})^*\partial_x M \arrow[r, squiggly, "(\mathcal{L}'_{\iota_{M_\perp}})^*\beta_x"] \arrow[ddrr, bend right, "\epsilon_x"'] & (\mathcal{L}'_{\iota_{M_\perp}})^*(\mathcal{L}'_{f_\perp+1})^*\partial_{f_\perp x}N \arrow[rr, squiggly, "(\mathcal{L}'_{\iota_{M_\perp}})^*(\mathcal{L}'_{f_\perp+1})^*i"] \arrow[d, dash, "\alpha"] \arrow[drr, phantom, "\Downarrow\nu"] & & (\mathcal{L}'_{\iota_{M_\perp}})^*(\mathcal{L}'_{f_\perp+1})^*\partial_{f_\perp y}N \arrow[r, squiggly, "(\mathcal{L}'_{\iota_{M_\perp}})^*(\beta_y)^{-1}"] \arrow[d, dash, "\alpha"] & (\mathcal{L}'_{\iota_{M_\perp}})^*\partial_y M \\
\arrow[r, phantom, "\sigma"] & (\mathcal{L}'_{f_\perp})^*(\mathcal{L}'_{\iota_{N_\perp}})^*\partial_{f_\perp x}N \arrow[rr, squiggly, "(\mathcal{L}'_{f_\perp})^*(\mathcal{L}'_{\iota_{N_\perp}})^*i"] \arrow[dr, dash, "(\mathcal{L}'_{f_\perp})^*(\epsilon_{f_\perp x})"'] \arrow[rr, phantom, bend right, "\Downarrow(\mathcal{L}'_{f_\perp})^*j"] & & (\mathcal{L}'_{f_\perp})^*(\mathcal{L}'_{\iota_{N_\perp}})^*\partial_{f_\perp y}N \arrow[r, phantom, "\sigma"] & \\
& & M \arrow[ur, dash, "(\mathcal{L}'_{f_\perp})^*(\epsilon^{-1}_{f_\perp y})"'] \arrow[uurr, bend right, "\epsilon_y^{-1}"'] & &
\end{tikzcd}
$$

Figure 3. Diagram for proof of Lemma 7.7

$$\mathsf{Section}(f) :\equiv \prod_{K:\mathcal{L}_\perp} \mathsf{Section}(f_\perp(K)) \times \mathsf{Section}(f'),$$

so we let $U_f :\equiv \left(\lambda K.V_{f_\perp(K)}, U_{f'}\right)$ where

$$V_{f_\perp(K)} : \mathsf{isEquiv}(f_\perp(K)) \to \mathsf{Section}(f_\perp(K))$$

is the evident forgetful function.

Now we show, by induction on n, that each such U_f is an equivalence. When $n :\equiv 0$, each U_f is a endofunction on $\mathbf{1}$, and so is an equivalence.

Now suppose that $n > 0$. We first construct a function F_f : $\mathsf{Section}(f) \to \mathsf{isIso}(f)$. Consider an element of $\mathsf{Section}(f)$: a section $s(K)$ of $f_\perp(K)$ for each $K : \mathcal{L}_\perp$ and a section s' of f'. Since M' is univalent, our inductive hypothesis applied to s' shows that f' is an isomorphism; thus it remains to show that each $f_\perp(K)$ is an equivalence.

Since $s(K)$ is a section of $f_\perp(K)$, it remains to show that $s(K)f_\perp(K)m = m$ for every $m : M_\perp(K)$. We have an identification $f_\perp(K)s(K)f_\perp(K)m = f_\perp(K)m$ and thus an isomorphism $i : f_\perp(K)s(K)f_\perp(K)m \cong f_\perp(K)m$. We have already shown that f' is an isomorphism $M' \cong (f_\perp)^*N'$, so by Proposition 5.7, we get an identification $M' = (\mathcal{L}'_{f_\perp})^*N'$. We can then apply Lemma 7.7 to get an isomorphism $s(K)f_\perp(K)m \cong m$. Since M is univalent, we then obtain an identification $s(K)f_\perp(K)m = m$.

Thus, given our $(\lambda K.s(K), s') : \mathsf{Section}(f)$, we have constructed an element of $\mathsf{isIso}(f)$; this defines $F_f : \mathsf{Section}(f) \to \mathsf{isIso}(f)$. Since $\mathsf{isIso}(f)$ is a proposition (by Lemma 5.5), $F_f U_f = 1$. Moreover, we constructed F_f and U_f in such a way that $U_f F_f = 1$.[2] Therefore, U_f is an equivalence.

Since $U_f : \mathsf{isIso}(f) \to \mathsf{Section}(f)$ is an equivalence, the forgetful function $(M \cong_L N) \to (M \twoheadrightarrow N)$ is also an equivalence. Using Proposition 5.7, we find that $\mathsf{idtovsm}$: $(M = N) \to (M \twoheadrightarrow N)$ is an equivalence. □

Example 7.9. A very split-surjective morphism between univalent FOLDS-categories is the same as a fully faithful and split (essentially) surjective functor between the corresponding categories. By [2, Lemma 6.6] this is the same as an equivalence of categories.

Such concrete examples suggest that it ought to suffice to use *very surjective* maps rather than very split-surjective ones. For instance, a functor between univalent 1-categories is an equivalence as soon as it is fully faithful and essentially surjective; split essential surjectivity is not required. One might hope to prove this in the general case by enhancing Lemma 7.7 to a "full-faithfulness" result that an induced map $f : (x \cong y) \to (fx \cong fy)$ is an equivalence; this would imply that a very surjective map between univalent structures is an embedding, and hence an equivalence. Unfortunately, we have not managed to do this; the problem is that it is not clear that an arbitrary morphism between structures induces any map at all on types of isomorphisms. In concrete cases, this is so because isomorphisms have an equivalent diagrammatic characterization by a "Yoneda lemma", but we do not know whether it is true in general.

8 Examples of signatures

In this section, we discuss some examples of signatures and their (univalent) structures.

Example 8.1 (First-order logic)**.** Consider a many-sorted first-order theory with relations and equality (assumed to be a congruence and equivalence relation):

$$
\begin{tikzcd}
E_1 \arrow[d, shift left] \arrow[d, shift right] & R_1 \arrow[dl] \arrow[dr] & E_2 \arrow[d, shift left] \arrow[d, shift right] & R_2 \arrow[dlll] \arrow[dr, shift left] \arrow[dr, shift right] & E_3 \arrow[d, shift left] \arrow[d, shift right] & R_3 \arrow[dl] \\
A_1 & & A_2 & & A_3 & \ldots
\end{tikzcd}
$$

- Univalence at E_i and R_i forces them to have at most one element each, i.e., to be relations.
- Univalence at A_i makes it a set whose equality is E_i.

We recover first-order logic with equality.

Note that any instance of this example, with sorts $(A_i)_{i:I}$, is also an instance of the SIP [11, Section 9.9] over the category Set^I—in particular, the examples of posets, monoids, groups, and fields mentioned in Section 1.1.

[2] Since we showed that $f_\perp(K)$ was an equivalence by making $s(K)$ a homotopy inverse of it, and U_f remembers not just the inverse map but one of the homotopies, we technically have to use here the fact that a homotopy inverse of a function g can be enhanced to an element of $\mathsf{isEquiv}(g)$ while changing at most one of the constituent homotopies.

Example 8.2 (†-categories)**.** A †-category is a category with coherent isomorphisms $(\_)^\dagger : \hom(x,y) \to \hom(y,x)$. A signature for †-categories is as follows,

$$\begin{tikzcd}
D \arrow[dr, shift left, "o"] \arrow[dr, shift right, "i"'] & T \arrow[d] \arrow[d, shift left] \arrow[d, shift right] & I \arrow[dl] & E \arrow[dll, shift left] \arrow[dll, shift right] \\
& A \arrow[d, shift right, "c"'] \arrow[d, shift left, "d"] & & \\
& O & &
\end{tikzcd}$$

with $co \overset{s}{=} di$ and $do \overset{s}{=} ci$ and the strict equalities of Figure 1. We assume that E is a congruence and an equivalence relation.

- Given a structure M, a FOLDS-isomorphism between $x, y : MO$ is a *unitary isomorphism* $f : x \cong y$, i.e., one satisfying $f^{-1} = f^\dagger$.
- In a univalent †-category, O is the groupoid of objects and unitary isomorphisms.
- Very split-surjective maps (which are underlying equivalences of univalent †-categories) are *†-equivalences*, involving unitary natural isomorphisms.

Further examples of higher structures that can be specified via a suitable signature are:

- a category with a presheaf on it;
- two categories and a(n ana)functor between them;
- two categories and a profunctor between them;
- a (bi)category and a displayed category (or a fibration) over it.

These examples will be explained in detail in an extended version of this article.

Example 8.3 (T_0-spaces)**.** We end by sketching some signatures whose structures include topological spaces, showing that our axiomatic signatures are more general than FOLDS-signatures. Since a topology is a structure on one underlying set, it suffices to consider height-2 signatures with $\mathcal{L}_\perp :\equiv \mathbf{1}$, with $\mathcal{L}' : \mathcal{U} \to \mathcal{U}$ remaining to be specified.

A first guess might be $\mathcal{L}'_M :\equiv (M \to \mathsf{Prop}_\mathcal{U})$, so that an $\mathcal{L}$-structure would be a type M with a predicate on its "type of subsets" $M \to \mathsf{Prop}_\mathcal{U}$ representing "is open". Unfortunately, this is not a covariant s-functor. We can make it covariant via direct images (using propositional truncation), but this is not *strictly* functorial, and moreover the resulting morphisms of structures would be open maps rather than continuous ones.

Covariant strict functoriality does hold, however, for the double-powerset functor $M \mapsto ((M \to \mathsf{Prop}_\mathcal{U}) \to \mathsf{Prop}_\mathcal{U})$, so we can use a definition of topological spaces that refers to sets of subsets instead of individual subsets. For instance, a topology is equivalent to a *convergence* relation between filters (which are sets of subsets) and points, hence can be regarded as a particular $\mathcal{L}$-structure with

$$\mathcal{L}'_M :\equiv ((M \to \mathsf{Prop}_\mathcal{U}) \to \mathsf{Prop}_\mathcal{U}) \times M.$$

The covariant functoriality specializes to the direct image of filters, so the $\mathcal{L}$-structure morphisms between topological spaces will be functions that preserve convergence, which is equivalent to continuity. Finally, univalence means that convergence is a proposition, that M is a set, and that two points are identified if exactly the same filters converge to them; the latter is an equivalent way of saying the space is T_0. Of course, not every $\mathcal{L}$-structure is a topological space; we could hope to single out the spaces with a "theory" in an appropriate "logic" based on our signatures.

We can also take $\mathcal{L}'_M :\equiv ((M \to \mathsf{Prop}_\mathcal{U}) \to \mathsf{Prop}_\mathcal{U})$ and associate a topological space M to the set of all sets of subsets $\mathcal{T}$ such that all open subsets belong to $\mathcal{T}$. Once again the structure morphisms between topological spaces are the continuous maps, and the univalent such structures are the T_0-spaces. Other topological structures such as uniform spaces and proximity spaces can similarly be represented as structures over suitable height-2 signatures.

9 Conclusion

We defined a general notion of isomorphism of objects within a categorical structure, yielding the companion notion of univalence of structure. Importantly, these notions only depend on the shape of the structures as specified by the signature, not on any axioms imposed on such structures (see also Remark 2.2). We then showed a Structure Identity Principle for univalent structures. This result specializes, for the signature of categories, to the fact that identifications between univalent categories correspond to equivalences between them.

Regarding the setting we have chosen for our work, it seems impossible to define a fully coherent notion of signature without 2LTT; it seems possible that a sufficiently-coherent "wild" notion might suffice for the particular results we prove here, but for further development of the theory it may prove necessary to have the fully coherent version. In addition, 2LTT is necessary for treating FOLDS-signatures of arbitrary height.

Here below we mention a few lines of inquiry that we would like to explore in the future:

- We would like to remove the splitness condition from Theorem 7.8, as discussed at the end of Section 7.
- We would like to define a formal syntax for axioms on a structure (cf. Section 2.2).
- We also aim to formalize the results presented here in a computer proof assistant implementing 2LTT.

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A Proofs for Section 2

Proof of Lemma 2.4. The underlying data of a type of objects with dependent hom-sets are the same. In one direction, we define $MI_x(f) :\equiv (f = 1_x)$ and $MT_{x,y,z}(f, g, h) :\equiv (h = g \circ f)$. In the other direction, we let 1_x be the unique $f : MA(x, x)$ such that $MI_x(f)$, and $g \circ f$ the unique h such that $MT_{x,y,z}(f, g, h)$. □

Proof of Theorem 2.12. First of all, since $f \cong g$ is a proposition, if the latter condition holds then each type $A(a, b)$ is a set. Thus, for the "if" direction, it will suffice to show that if $E_{a,b}(f, g)$ then $f \cong g$. However, this follows easily from the axioms that E is a congruence for T and I.

For the "only if" direction, we must show that $(f \cong g) \Rightarrow (f = g)$ in a FOLDS-precategory. However, since $E_{a,b}(f, f)$ always, $f \cong g$ implies in particular that $E_{a,b}(f, g)$, hence $f = g$ by standardness. □

B Proofs for Section 4

Proof of Proposition C.7. Consider $\mathcal{L}, \mathcal{M}, \mathcal{N} : \mathsf{FSig}(p)$ and $F : \hom_{\mathsf{FSig}(p)}(\mathcal{L}, \mathcal{M}), F : \hom_{\mathsf{FSig}(p)}(\mathcal{M}, \mathcal{N})$. Let $(G \circ F)(n) :\equiv G(n) \circ F(n)$ and $(G \circ F)_{x,y} :\equiv G_{F(n)(x),F(n)(y)} \circ F_{x,y}$. It is clear that this forms a strict semi-functor. It forms a discrete opfibration since for every $n : \mathbb{N}^s_{<p}, K : \mathcal{L}(n)$, $G \circ F : \mathsf{Fanout}_n(K) \to \mathsf{Fanout}_n(GFK)$ is the composition of the isomorphisms $F : \mathsf{Fanout}_n(K) \to \mathsf{Fanout}_n(FK)$ and $G : \mathsf{Fanout}_n(FK) \to \mathsf{Fanout}_n(GFK)$.

For any $\mathcal{L} \in \mathsf{FSig}(p)$, there is an identity discrete opfibration $1_{\mathcal{L}} : \hom_{\mathsf{FSig}(p)}(\mathcal{L}, \mathcal{L})$ whose components are all identity functions.

The composition defined above is clearly associative and left and right unital. □

C FOLDS-signatures and translation to axiomatic signatures

Definition C.1. Let $p : \mathbb{N}^s$. A **one-way semi-category**[3] $\mathcal{L}$ **of height** p consists of

- A family of types of objects $\mathcal{L} : \mathbb{N}^s_{<p} \to \mathcal{U}^s$. We call $\mathcal{L}(n)$ the family of *sorts of rank n*.
- A family of types of morphisms
$$\hom_{\mathcal{L}} : \textstyle\prod_{(n:\mathbb{N}^s_{<p})} \prod_{(m:\mathbb{N}^s_{<n})} \mathcal{L}(n) \to \mathcal{L}(m) \to \mathcal{U}^s.$$
- A suitably typed composition operation on morphisms
$$(\cdot) : \hom_{\mathcal{L}}(x, y) \to \hom_{\mathcal{L}}(y, z) \to \hom_{\mathcal{L}}(x, z)$$
that is strictly associative: $f \cdot (g \cdot h) \overset{s}{=} (f \cdot g) \cdot h$.

The objects of $\mathcal{L}$ denote *sorts* and the morphisms *dependencies*. Morphisms can only go "downwards", that is, a sort can only depend on sorts "below" it; see Figure 1.

Definition C.2. Given a one-way semi-category $\mathcal{L}$, the **fan-out type of** $K : \mathcal{L}(n)$ **at** $m < n$ is
$$\mathsf{Fanout}_m(K) :\equiv \sum_{L:\mathcal{L}(m)} \hom_{\mathcal{L}}(K, L).$$

Definition C.3. A **FOLDS-signature of height** p is a one-way semi-category $\mathcal{L}$ of height p for which each $\mathcal{L}(n)$ is fibrant and each $\mathsf{Fanout}_m(K)$ is cofibrant. The type of FOLDS-signatures of height p is denoted by $\mathsf{FSig}(p)$.

Examples C.4. We denote the FOLDS-signatures shown in Figure 1 by $\mathcal{L}_{\mathrm{rg}}$, $\mathcal{L}_{\mathrm{cat}}$, and $\mathcal{L}_{\mathrm{cat+E}}$ respectively. At each rank we have a finite-fibrant type of objects of that rank.

Definition C.5. For $\mathcal{L}, \mathcal{M} : \mathsf{FSig}(p)$, a **strict semi-functor** $F : \mathcal{L} \to \mathcal{M}$ consists of functions

- $F : \prod_{(n:\mathbb{N}^s_{<p})} \mathcal{L}(n) \to \mathcal{M}(n)$
- $F : \prod_{(m:\mathbb{N}^s_{<p})} \prod_{(n:\mathbb{N}^s_{<m})} \prod_{(x:\mathcal{L}(m))} \prod_{(y:\mathcal{L}(n))} \hom_{\mathcal{L}}(x, y) \to \hom_{\mathcal{M}}(Fx, Fy)$

which strictly preserves composition.

Note that we take the ranks of objects to be specified data, which are preserved strictly by strict semi-functors. In particular, a strict semi-functor $F : \mathcal{L} \to \mathcal{M}$ induces a function $F_m(K) : \mathsf{Fanout}_m(K) \to \mathsf{Fanout}_m(FK)$ for every $K : \mathcal{L}(n)$ and $m < n$. We sometimes write F instead of $F_m(K)$ when no confusion can arise.

[3] Makkai uses honest categories, but any one-way category is freely generated by a semi-category, so it is simpler to leave out the identity morphisms.

Definition C.6. A strict semi-functor $F : \mathcal{L} \to \mathcal{M}$ is a **discrete opfibration** when all the functions

$$F_m(K) : \mathsf{Fanout}_m(K) \to \mathsf{Fanout}_m(FK)$$

are isomorphisms. Let $\hom_{\mathsf{FSig}(p)}(\mathcal{L}, \mathcal{M})$ denote the type of such discrete opfibrations.

We can think of each $\mathsf{Fanout}_m(K)$ as the structure on which K depends, and then discrete opfibrations are exactly those functors which preserve this dependency structure.

Proposition C.7. *The type* $\mathsf{FSig}(p)$*, the types* $\hom_{\mathsf{FSig}(p)}(\mathcal{L}, \mathcal{M})$ *given by discrete opfibrations, and the obvious composition and identity form an s-category.*

C.1 Translation from FOLDS-signatures to axiomatic signatures

A FOLDS-signature $\mathcal{L}$ will be mapped to an axiomatic signature whose $\perp$ is $\mathcal{L}(0)$, so we define:

Notation C.8. For $p > 0$, $\mathcal{L} : \mathsf{FSig}(p)$, let $\mathcal{L}_\perp :\equiv \mathcal{L}(0)$.

Now we formally define the derivative operation on FOLDS-signatures discussed above.

Definition C.9. Consider $p > 0$, $\mathcal{L} : \mathsf{FSig}(p)$, and $M : \mathcal{L}_\perp \to \mathcal{U}$. The **derivative of $\mathcal{L}$ with respect to** M is the one-way semi-category $\mathcal{L}'_M$ of height $p-1$ with objects and morphisms defined as follows:

- $\mathcal{L}'_M(n) :\equiv \sum_{K:\mathcal{L}(n+1)} \prod_{(F:\mathsf{Fanout}_0(K))} M(\pi_1 F)$
- $\hom_{\mathcal{L}'_M}((K_1, \alpha_1), (K_2, \alpha_2)) :\equiv$
  $\sum_{(f:\hom(K_1,K_2))} \prod_{(F:\mathsf{Fanout}_0(K_2))} \alpha_1(F \circ f) \overset{s}{=} \alpha_2(F)$

where $\pi_1 : \mathsf{Fanout}_0(K) \to \mathcal{L}_\perp$ is the projection and $F \circ f$ denotes the function $\mathsf{Fanout}_0(K_2) \to \mathsf{Fanout}_0(K_1)$ given by precomposition.

Example C.10. If $p \equiv 1$ then $\mathcal{L}_{>0}$ is empty. Thus, no matter what $M : \mathcal{L}_\perp \to \mathcal{U}$ we choose, $\mathcal{L}'_M$ is the empty signature.

Example C.11. If $\mathcal{L}$ is a signature of height 2, then it consists of two types of sorts $\mathcal{L}(0)$ and $\mathcal{L}(1)$ and a family of hom-types $\hom_{\mathcal{L}} : \mathcal{L}(1) \to \mathcal{L}(0) \to \mathcal{U}^s$. Then for any $M : \mathcal{L}(0) \to \mathcal{U}$, the signature $\mathcal{L}'_M$ has height 1 consisting of just a single type of sorts of rank 0. Each such sort is, by definition, a sort $K : \mathcal{L}(1)$ in $\mathcal{L}$ of rank 1 together with, for any $L : \mathcal{L}(0)$ and $g : \hom_{\mathcal{L}}(K, L)$, an element of ML.

Example C.12. We have $\mathcal{L}_{\mathrm{rg}_\perp} \equiv \{O\}$. Let M be (a function picking out) the two-element set $\{a, b\}$. Then $(\mathcal{L}_{\mathrm{rg}})'_M$ is the following signature, with four sorts of rank 0 and two sorts of rank 1:

$$\begin{tikzcd}
1 & I(a,a) \arrow[d, "i"] & & & I(b,b) \arrow[d, "i"] \\
0 & A(a,a) & A(a,b) & A(b,a) & A(b,b)
\end{tikzcd}$$

The extra conditions on FOLDS-signatures in Definition C.3 ensure that these signatures are closed under derivation:

Proposition C.13. *For $p > 0$, $\mathcal{L} : \mathsf{FSig}(p)$, and $M : \mathcal{L}_\perp \to \mathcal{U}$, the one-way semi-category $\mathcal{L}'_M$ is a FOLDS-signature.*

Proof of Proposition C.13. Since each $\mathsf{Fanout}_0(K)$ is cofibrant and each $M(\pi_1 F)$ is fibrant, we have that $\prod_{(F:\mathsf{Fanout}_0(K))} M(L)$ is fibrant. Since $\mathcal{L}(n+1)$ is fibrant, so is

$$\sum_{(K:\mathcal{L}(n+1))} \prod_{(F:\mathsf{Fanout}_0(K))} M(\pi_1 F).$$

Now consider $n : \mathbb{N}^s_{<p}$, $m : \mathbb{N}^s_{<n}$, and $(K, \alpha) : \mathcal{L}'_M(n)$, We have

$$\begin{aligned}
&\mathsf{Fanout}_m(K, \alpha) \\
&\equiv \sum_{(L,\beta):\mathcal{L}'_M(m)} \hom_{\mathcal{L}'_M}(K, L) \\
&\cong \sum_{\substack{L:\mathcal{L}(m+1) \\ \beta:\prod_{(\mathsf{Fanout}_0(L))} M(L) \\ f:\hom(K,L)}} \prod_{(N,g):\mathsf{Fanout}_0(L)} \alpha(N, gf) \overset{s}{=} \beta(N, g) \\
&\cong \sum_{\substack{G:\mathsf{Fanout}_{m+1}(K) \\ \beta:\prod_{(\mathsf{Fanout}_0(L))} M(L)}} \prod_{(N,g):\mathsf{Fanout}_0(L)} \alpha(N, gf) \overset{s}{=} \beta(N, g) \\
&\cong \sum_{G:\mathsf{Fanout}_{m+1}(K)} \mathbf{1} \\
&\cong \mathsf{Fanout}_{m+1}(K)
\end{aligned}$$

Here, we expand $\mathcal{L}'_M(m)$ and $\hom_{\mathcal{L}'_M}(K, L)$ to get the first isomorphism. We rearrange pairs and use the definition of $\mathsf{Fanout}_{m+1}(K)$ to get the second isomorphism. To get the third, observe that

$$\sum_{(\beta:\prod_{(\mathsf{Fanout}_0(L))} M(L))} \prod_{((N,g):\mathsf{Fanout}_0(L))} \alpha(N, gf) \overset{s}{=} \beta(N, g)$$

is isomorphic to $\mathbf{1}$.

Since $\mathsf{Fanout}_{m+1}(K)$ is cofibrant, so is $\mathsf{Fanout}_m(K, \alpha)$. □

Definition C.14. Consider $p : \mathbb{N}^s$, $\mathcal{L}, \mathcal{M} : \mathsf{FSig}(p)$, $H : \hom_{\mathsf{FSig}(p)}(\mathcal{L}, \mathcal{M})$, $L : \mathcal{L}_\perp \to \mathcal{U}$, $M : \mathcal{M}_\perp \to \mathcal{U}$, and $h : \prod_{(K:\mathcal{L}_\perp)} LK \to MH_\perp K$.

We define the functor $H'_h : \mathcal{L}'_L \to \mathcal{M}'_M$ as follows.

- Consider an $n : \mathbb{N}^s_{<p-1}$ and a $(K, \alpha) : \mathcal{L}'_L(n)$, (so $\alpha : \prod_{(F:\mathsf{Fanout}_K(0))} L(\pi_1 F)$). We define $H'_h(K, \alpha) : \mathcal{M}'_M(n)$ to be $(H(K), \beta)$, where

  $$\beta(F) :\equiv h_{\pi_1(H^{-1}F)}\big(\alpha(H^{-1}F)\big) : MH_\perp(\pi_1 H^{-1} F) \overset{s}{=} M(\pi_1 F)$$

  for $F : \mathsf{Fanout}_0(HK)$.
- Consider a morphism $(f, \phi) : \hom_{\mathcal{L}'_L}((K_1, \alpha_1), (K_2, \alpha_2))$. We define $H'_h(f, \phi) : \hom_{\mathcal{M}'_M}((HK_1, \beta_1), (HK_2, \beta_2))$ to be (Hf, ψ). To define ψ on $F : \mathsf{Fanout}_0(HK_2)$, we must check that

  $$h_{\pi_1(H^{-1}(F \circ Hf)}(\alpha_1(H^{-1}(F \circ Hf))) \overset{s}{=} h_{\pi_1(H^{-1}F)}(\alpha_2(H^{-1}F)).$$

  But we have $H^{-1}(F \circ Hf) \overset{s}{=} H^{-1}(F) \circ f$ (since applying the isomorphism H produces $F \circ Hf$ on both sides) and $\alpha_1(H^{-1}(F) \circ f) \overset{s}{=} \alpha_2(H^{-1}(F))$ by ϕ.

Lemma C.15. *We have that*

1. *for $\mathcal{L} : \mathsf{FSig}(p)$*

$$(1_{\mathcal{L}})'_{\lambda x.1_{Lx}} \overset{s}{=} 1_{\mathcal{L}'_L}$$

2. *for $\mathcal{L}, \mathcal{M}, \mathcal{N} : \mathsf{FSig}(p)$, $H : \mathrm{hom}_{\mathsf{FSig}(p)}(\mathcal{L}, \mathcal{M})$, $I : \mathrm{hom}_{\mathsf{FSig}(p)}(\mathcal{M}, \mathcal{N})$, $L : \mathcal{L}_\perp \to \mathcal{U}$, $M : \mathcal{M}_\perp \to \mathcal{U}$, $N : \mathcal{N}_\perp \to \mathcal{U}$, $h : \prod_{(x:\mathcal{L}_\perp)} Lx \to MH_\perp x$, and $i : \prod_{(x:\mathcal{M}_\perp)} Mx \to NI_\perp x$,*

$$I'_i \circ H'_h \overset{s}{=} (I \circ H)'_{i \circ h}$$

where $i \circ h(x, \ell) :\equiv i(H_\perp x, h(x, \ell))$.

Proof. Note that the desired strict equalities are obvious on the first components of objects and morphisms in $\mathcal{L}'_L$. The desired strict equalities on the second components of morphisms follow from UIP and function extensionality. Thus, we check the strict equalities just on the second components of objects.

To check the strict equality (1) on objects, observe that

$$\begin{aligned}\pi_2(1_{\mathcal{L}})'_{1_L}(K, \alpha) &\overset{s}{=} \lambda F.\lambda x.1_{Lx}(\pi_1(1_{\mathcal{L}}^{-1}F), \alpha(1_{\mathcal{L}}^{-1}F)) \\ &\overset{s}{=} \alpha.\end{aligned}$$

To check the strict equality (2) on objects, calculate that

$$\begin{aligned}&\left(\pi_2(I \circ H)'_{i\circ h}(K, \alpha)\right)(F) \\ &\overset{s}{=} ih(\pi_1((IH)^{-1}F), \alpha((IH)^{-1}F)) \\ &\overset{s}{=} i(H_\perp \pi_1((IH)^{-1}F), h(\pi_1((IH)^{-1}F), \alpha((IH)^{-1}F))) \\ &\overset{s}{=} i(\pi_1(I^{-1}F), h(\pi_1(H^{-1}I^{-1}F), \alpha(H^{-1}I^{-1}F))) \\ &\overset{s}{=} \left(\pi_2(I'_i \circ H'_h)(K, \alpha)\right)(F).\end{aligned}$$

□

Lemma C.16. *Consider $\mathcal{L} : \mathsf{FSig}(p)$ and $M : \mathcal{L}_\perp \to \mathcal{U}$. Let $\mathcal{L}_{>0}$ be the one-way semi-category given by $\mathcal{L}_{>0}(n) :\equiv \mathcal{L}_{>0}(n+1)$ and $\mathrm{hom}_{\mathcal{L}_{>0}}(x, y) :\equiv \mathrm{hom}_{\mathcal{L}}(x, y)$.*

The evident forgetful functor $U : \mathcal{L}'_M \to \mathcal{L}_{>0}$ is a discrete opfibration.

Proof. Consider a $(K, \alpha) : \mathcal{L}'_M$ where $K : \mathcal{L}(n+1)$, $\alpha : \prod_{(F:\mathsf{Fanout}_0(K))} M(\pi_1 F)$ and a $(L, f) : \mathsf{Fanout}_m(K)$ where $L : \mathcal{L}(m)$, $f : \mathrm{hom}(K, L)$. Let $U^{-1}(L, f)$ be $((L, \beta), (f, \gamma))$ where we define $\beta : \prod_{(F:\mathsf{Fanout}_0(L))} M(\pi_1 F)$ and $\gamma : \prod_{(F:\mathsf{Fanout}_0(L))} \alpha(F \circ f) \overset{s}{=} \beta(F)$ as follows. Let $\beta(F) :\equiv \alpha(F \circ f)$. Then we have γ by construction.

Clearly, $UU^{-1} \overset{s}{=} 1_{\mathsf{Fanout}_m(K)}$. To show

$$U^{-1}U \overset{s}{=} 1_{\mathsf{Fanout}_m((K,\alpha))} \quad ,$$

consider a $((L, \beta), (f, \gamma)) : \mathsf{Fanout}_m((K, \alpha))$. We get that

$$U^{-1}U((L, \beta), (f, \gamma)) \overset{s}{=} ((L, \beta'), (f, \gamma'))$$

and

$$\gamma^{-1} * \gamma' : \prod_{F:\mathsf{Fanout}_0(L)} \beta(F) \overset{s}{=} \beta'(F)$$

By function extensionality, $\beta \overset{s}{=} \beta'$ and by UIP and function extensionality, $\gamma \overset{s}{=} \gamma'$. □

Lemma C.17. *Consider $\mathcal{L}, \mathcal{M}, \mathcal{N} : \mathsf{FSig}(p)$. Consider functors F from $\mathcal{L}$ to $\mathcal{M}$ and G from $\mathcal{M}$ to $\mathcal{N}$ such that $G \circ F$ and G are discrete opfibrations. Then F is a discrete opfibration.*

Proof. Consider a $K : \mathcal{L}(n)$. The following strictly commutative diagram shows that F is an isomorphism on fanouts, and thus a discrete opfibration.

$$\begin{tikzcd} \mathsf{Fanout}_m(K) \arrow[r, "F"] \arrow[dr, "\cong", "G \circ F"'] & \mathsf{Fanout}_m(FK) \arrow[d, "\cong"', "G"] \\ & \mathsf{Fanout}_m(GFK) \end{tikzcd}$$

□

Proposition C.18. *The functor $H'_h : \mathcal{L}'_L \to \mathcal{M}'_M$ from Definition C.14 is a discrete opfibration.*

Proof of Proposition C.18. The following square commutes.

$$\begin{tikzcd} \mathcal{L}'_L \arrow[r, "H'_h"] \arrow[d, "U"'] & \mathcal{M}'_M \arrow[d, "U"] \\ \mathcal{L}_{>0} \arrow[r, "H_{>0}"'] & \mathcal{M}_{>0} \end{tikzcd}$$

Note that since H is a discrete opfibration, so is $H_{>0}$. Since both instances of U are also discrete opfibrations (Lemma C.16), we find (using Lemma C.17) that H'_h is a discrete opfibration. □

We now have all the ingredients for our translation.

Theorem C.19. *For each $p : \mathbb{N}^s$, define an s-functor $E_p : \mathsf{FSig}(p) \to \mathsf{Sig}(p)$ by induction on p as follows.*

Since $\mathsf{Sig}(0)$ is the trivial category on $\mathbf{1}$, there is a unique s-functor $\mathsf{FSig}(0) \to \mathsf{Sig}(0)$ (which is actually an equivalence).

For $p > 0$ and $\mathcal{L} : \mathsf{FSig}(p)$, let $E_p(\mathcal{L})$ consist of:

1. *The type $\mathcal{L}_\perp : \mathcal{U}$.*
2. *The functor $E_{p-1}\mathcal{L}'_{-} : (\mathcal{L}_\perp \to \mathcal{U}) \to \mathsf{Sig}(p-1)$ defined on objects as in Definition C.9 and Proposition C.13 and on morphisms as in Definition C.14 and Proposition C.18.*

For $\mathcal{L}, \mathcal{M} : \mathsf{FSig}(p)$ and $F : \mathrm{hom}(\mathcal{L}, \mathcal{M})$, let $E_p(F)$ consist of:

1. *The function $F_\perp : \mathcal{L}_\perp \to \mathcal{M}_\perp$.*
2. *The natural transformation with underlying function*

$$E_{p-1}F'_{\lambda x.1_{-x}} : \prod_{M:\mathcal{M}_\perp \to \mathcal{U}} \mathrm{hom}(\mathcal{L}'_{M \circ F_\perp}, \mathcal{M}'_M)$$

defined in Definition C.14 and Proposition C.18.

Proof of Theorem C.19. We check that E_p is functorial.

For any $\mathcal{L} : \mathsf{FSig}(p)$, we have the following.

$$\begin{aligned}\pi_1 E_p(1_{\mathcal{L}}) &\overset{s}{=} 1_{\mathcal{L}_\perp} \\ &\overset{s}{=} \pi_1(1_{E_p \mathcal{L}}) \\ \pi_2 E_p(1_{\mathcal{L}}) &\overset{s}{=} E_{p-1} \circ 1_{\mathcal{L}}{}'_{\lambda x.1_{-x}}\end{aligned}$$

$$\begin{aligned} &\overset{\text{s}}{=} 1_{E_{p-1}1_{\mathcal{L}'_{\lambda x.1-x}}} \\ &\overset{\text{s}}{=} \pi_2(1_{E_p\mathcal{L}}) \end{aligned}$$

For any $\mathcal{M}, \mathcal{N}, \mathcal{P} : \mathsf{FSig}(p), F : \hom(\mathcal{M}, \mathcal{N}), G : \hom(\mathcal{N}, \mathcal{P})$, we have the following.

$$\begin{aligned} \pi_1 E_p(G \circ F) &\overset{\text{s}}{=} (G \circ F)_\perp \\ &\overset{\text{s}}{=} G_\perp \circ F_\perp \\ &\overset{\text{s}}{=} \pi_1(E_p G \circ E_p F) \end{aligned}$$

$$\begin{aligned} \left(\pi_2 E_p(G \circ F)\right)(M) &\overset{\text{s}}{=} E_{p-1} \circ (G \circ F)'_{\lambda x.1_{Mx}} \\ &\overset{\text{s}}{=} E_{p-1}(G'_{\lambda x.1_{Mx}} \circ F'_{\lambda x.1_{Mx}}) \\ &\overset{\text{s}}{=} (E_{p-1}G'_{\lambda x.1_{Mx}}) \circ (E_{p-1}F'_{\lambda x.1_{Mx}}) \\ &\overset{\text{s}}{=} \pi_2 E_{p-1} G \circ \pi_2 E_{p-1} F. \end{aligned}$$ □

Intuitively, this can be thought of as mapping into the s-category *coinductively* defined by a derivative functor, with the result landing inside the inductive part (our axiomatic signatures) because our FOLDS-signatures have finite height.