Type theory and concurrency: directed homotopy type theory

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Goal

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To develop a logic for directed homotopy theory

- To prove theorems about concurrent processes

Homotopy type theory

HoTT

A logic for homotopy theory.

Basic objects of the language

Dependent types:

$$b : B \vdash E(b) \ \textsc{type} \quad \rightsquigarrow \quad \begin{array}{c} E \\ \Big\downarrow \text{fibration } \pi \\ B \end{array}$$

Dependent terms:

$$b : B \vdash e(b) : E(b) \quad \rightsquigarrow \quad \begin{array}{c} E \\ \text{section } e \Big\uparrow \Big\downarrow \text{fibration } \pi \\ B \end{array}$$

Homotopy type theory

Type formers

Usually postulate that certain types can be formed:
e.g., from $b : B \vdash E(b)$ and $b : B \vdash F(b)$ can form

- the coproduct $b : B \vdash E(b) + F(b)$,
- the hom-type $b : B \vdash E(b) \to F(b)$.

As in category theory: we postulate that objects with certain universal properties exist.

The surprising type former

The identity type

Equality is defined in this language by a universal property:

- For $b : B \vdash E(b)$, we define a new type $b : B \vdash \mathsf{Id}_b(E)$ to be the *smallest reflexive relation* on E.

$$b : B \vdash \epsilon_0 \times \epsilon_1 : \mathsf{Id}_b(E) \to E(b) \times E(b)$$
$$b : B \vdash r(b) : E(b) \to \mathsf{Id}_b(E)$$

- We think of terms $b : B \vdash p(b) : \mathsf{Id}_b(E)$ as being an equality $\epsilon_0(pb) = \epsilon_1(pb)$

This makes equality *weaker* than usual set-theoretic equality.

- Easier for a computer to handle.
- The definition says that $\mathsf{Id}_b(E)$ is a (very good) path object:

$$E(b) \xrightarrow[\text{acof}]{r} \mathsf{Id}_b(E) \xrightarrow[\text{fib}]{\epsilon_0 \times \epsilon_1} E(b) \times E(b)$$

Why HoTT?

- It's a logic for homotopy theory.

Theorem (N)

Let $\mathcal{C}$ be a finitely complete category.

The category of models[1] of type theory with Id types in $\mathcal{C}$ is equivalent to the category of weak factorization systems $(\mathcal{L}, \mathcal{R})$ on $\mathcal{C}$ where

1. every map to the terminal object is in $\mathcal{R}$ and
2. $\mathcal{L}$ is stable under pullback along $\mathcal{R}$.

[1] display map categories

Why HoTT?

- It's a logic for homotopy theory.
- Which can be used for formal verification on a computer.

Example: reversing paths

Thm: Paths are reversible.

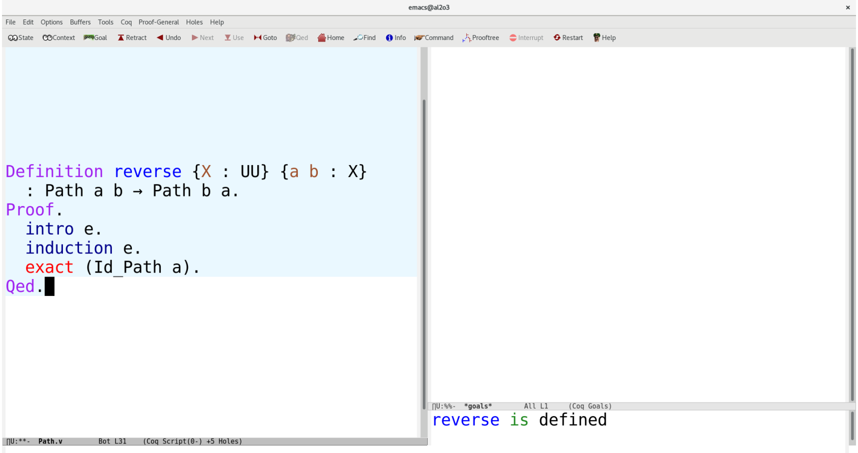
A categorical proof: Given a very good path object

$$X \xrightarrow[\text{acof}]{r} \mathrm{Path}(X) \xrightarrow[\text{fib}]{\epsilon_0 \times \epsilon_1} X \times X \ ,$$

we get a solution to the following lifting problem.

$$\begin{array}{ccc} X & \xrightarrow{r} & \mathrm{Path}(X) \\ {\scriptstyle r}\downarrow & \nearrow & \downarrow{\scriptstyle \epsilon_0 \times \epsilon_1} \\ \mathrm{Path}(X) & \xrightarrow{\epsilon_1 \times \epsilon_0} & X \times X \end{array}$$

A type theoretic proof in the computer:



Why HoTT?

- It's a logic for homotopy theory.
- Which can be used for formal verification on a computer.
- Everything is invariant under homotopy:
  for every $b : B \vdash E(b)$, for every $a, b : B$, there is a function

$$\text{transport} : E(a) \times \text{Id}(a, b) \to E(b).$$

Example: contractibility

Given $b : B \vdash E(b)$, define $b : B \vdash \text{isContr}(Eb)$ (type of contractions of Eb).

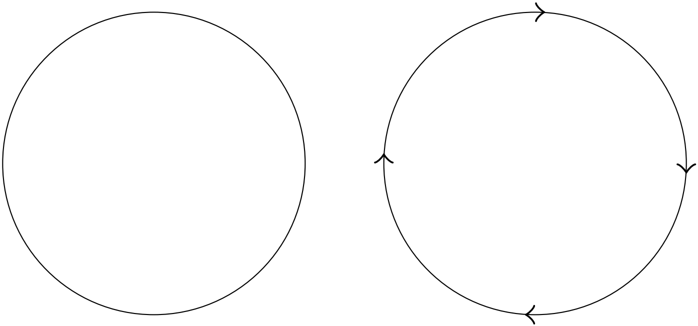
If there is a contraction $c : \text{isContr}(Ea)$ and a path $p : \text{Id}(a, b)$, then there is a contraction $\text{transport}(c, p) : \text{isContr}(Eb)$.

Even true for $T : \text{TYPE} \vdash \text{isContr}(T)$.

Directed spaces

Rough definition

A directed space is a space together with a subset of its paths which are marked as *directed*.



Directed spaces and concurrency

Concurrent processes can be represented by directed spaces.

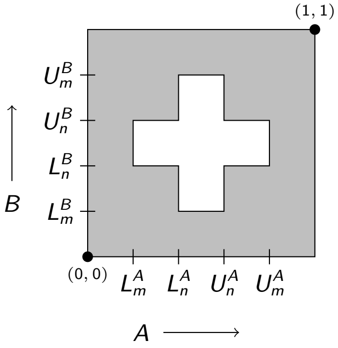


Figure: The Swiss flag

Fundamental question:

Which states are safe? Which states are reachable?

Design criteria for directed HoTT

- A logic for directed homotopy theory
  - Same underlying syntax as HoTT
  - With a new type former $\mathrm{hom}(-)$ whose terms represent directed paths
- Which can be used for formal verification on a computer.
- Where everything is covariant with directed homotopy: for every $b : B \vdash E(b)$, for every $a, b : B$, there is a function

$$\mathrm{transport} : E(a) \times \mathrm{hom}(a, b) \to E(b).$$

Example: reachability

Given $* \vdash D$, define $d : D \vdash \mathrm{Reach}(d)$ (type of ways to reach d from the initial point).

If there is a $r : \mathrm{Reach}(d)$ and a directed path $p : \mathrm{hom}(d, e)$, then there is a $\mathrm{transport}(r, p) : \mathrm{Reach}(e)$.

Rules for directed HoTT

$$\frac{\Gamma \vdash T : \mathcal{U}}{\Gamma \vdash T^{\mathsf{core}} : \mathcal{U}} \qquad \frac{\Gamma \vdash T : \mathcal{U}}{\Gamma \vdash T^{\mathsf{op}} : \mathcal{U}} \qquad \frac{\Gamma \vdash T : \mathcal{U} \qquad \Gamma \vdash t : T^{\mathsf{core}}}{\Gamma \vdash \iota t : T} \qquad \frac{\Gamma \vdash T : \mathcal{U} \qquad \Gamma \vdash t : T^{\mathsf{core}}}{\Gamma \vdash \iota^{\mathsf{op}} t : T^{\mathsf{op}}}$$

$$\frac{\Gamma \vdash T : \mathcal{U} \qquad \Gamma \vdash s : T^{\mathsf{op}} \qquad \Gamma \vdash t : T}{\Gamma \vdash \mathtt{hom}_T(s, t) : \mathcal{U}} \qquad \frac{\Gamma \vdash T : \mathcal{U} \qquad \Gamma \vdash t : T^{\mathsf{core}}}{\Gamma \vdash 1_t : \mathtt{hom}_T(\iota^{\mathsf{op}} t, \iota t) : \mathcal{U}}$$

$$\frac{\Gamma \vdash T : \mathcal{U} \qquad \Gamma, s : T^{\mathsf{core}}, t : T, f : \mathtt{hom}_T(\iota^{\mathsf{op}} s, t) \vdash D(f) : \mathcal{U} \qquad \Gamma, s : T^{\mathsf{core}} \vdash d(s) : D(1_s)}{\Gamma, s : T^{\mathsf{core}}, t : T, f : \mathtt{hom}_T(\iota^{\mathsf{op}} s, t) \vdash \delta_r(d, f) : D(f)}$$

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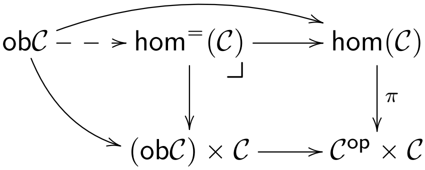
A model in categories

- There is a weak factorization system $(\mathcal{L}, \mathcal{R})$ in *Cat* enrichedly cofibrantly generated by the inclusion of the domain into a morphism.

$$0 \hookrightarrow (0 \to 1)$$

- The usual functor hom : $\mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to$ *Set* becomes a Grothendieck opfibration $\pi : \mathrm{hom}(\mathcal{C}) \to \mathcal{C}^{\mathsf{op}} \times \mathcal{C}$, which is in $\mathcal{R}$.
- There is a functor $\mathrm{ob}\mathcal{C} \to \mathrm{hom}^{=}(\mathcal{C})$ which is in $\mathcal{L}$.



- The rules above axiomatize the lifitng property that these maps (and other maps of the wfs) have against each other.

Summary & future work

Summary

We have:

- a type theory for directed paths
- with a model in *Cat*.

Summary

We need:

- to integrate this into old HoTT (i.e. have Id and hom in the same theory);
- to find a model in a category of directed spaces;
- to find models in all categories of directed spaces.

Thank you!