The enriched Curry-Howard-Lambek correspondence

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Outline

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Internal languages

Type theories give us internal languages for various kinds of categorical structures.

simply typed lambda calculus	$\leftrightsquigarrow$ [1]	cartesian closed categories
Martin-Löf type theory	$\leftrightsquigarrow$	lcc categories with fibrations
linear type theory	$\leftrightsquigarrow$	monoidal (*-autonomous) categories
bicategorical type theory	$\leftrightsquigarrow$	bicategories
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	$\leftrightsquigarrow$	enriched categories

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enriched simply typed lambda calculus	$\leftrightsquigarrow$	enriched categories

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Enriched categories

Definition

An *enrichment* of a category $\mathcal{C}$ in a monoidal category $\mathcal{V}$ consists of

- a functor $\underline{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{V}$
- a morphism $\mathbb{I} \to \underline{\mathcal{C}}(A, A)$ for each $A \in \mathrm{ob}\ \mathcal{C}$
- a morphism $\underline{\mathcal{C}}(A, B) \otimes \underline{\mathcal{C}}(B, C) \to \underline{\mathcal{C}}(A, C)$ for $A, B, C \in \mathrm{ob}\ \mathcal{C}$
- an isomorphism $\mathcal{V}(\mathbb{I}, \underline{\mathcal{C}}(A, B)) \cong \mathcal{C}(A, B)$ for $A, B \in \mathrm{ob}\ \mathcal{C}$.

such that ...

Enriched categories

Enriched categories are everywhere!

- 2-category (category enriched in categories) of categories
- simplicial sets are enriched in themselves (true for any topos)
- (Lawvere) metric spaces are categories enriched in the real numbers
- dg-categories are enriched in chain complexes
- additive, triangulated, abelian categories
- ...

Simply typed lambda calculus

Judgments:

- Γ ctx (list of types e.g., $x : R, y : S, z : T$)
- T type
- $\Gamma \vdash t : T$
- $S = T$ type
- $\Gamma \vdash s = t : T$

Under the logical interpretation (Curry-Howard correspondence),

- types correspond to propositions,
- terms $\Gamma \vdash t : T$ correspond to derivations of T from Γ.

Under the categorical interpretation (Lambek correspondence),

- types correspond to objects in a category,
- terms $\Gamma \vdash t : T$ correspond to morphisms $\Gamma \to T$.

Simply typed lambda calculus

Type formers:

- $S \times T$
- $S \to T$

Under the logical interpretation (Curry-Howard correspondence),

- $\times$ corresponds to $\wedge$
- $\to$ corresponds to $\Rightarrow$

Under the categorical interpretation (Lambek correspondence),

- $\times$ corresponds to the cartesian product
- $\to$ corresponds to internal hom / exponentiation (making the category cartesian closed)

Outline

Motivation and background

Enriched Lambek

Enriched Curry-Howard

Approach

Conundrum:

- Syntax naturally forms a collection of sets
- But enriched categories generalize sets $\rightsquigarrow$ objects in an arbitrary category

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Conundrum:

- Syntax naturally forms a collection of sets
- But enriched categories generalize sets $\rightsquigarrow$ objects in an arbitrary category
- We take a middle road, and focus on presheaves

Shape-based approach

When we study a 2-category $\mathcal{C}$, we want to know about

- 1-cells: $\hom_{\mathcal{C}} at(*, \hom_{\mathcal{C}}(A, B))$
- 2-cells: $\hom_{\mathcal{C}} at(\rightarrow, \hom_{\mathcal{C}}(A, B))$

When we study a simplicial set-category $\mathcal{C}$, we want to know about

- 1-cells: $\hom_{s\mathcal{S}et}(\Delta^0, \hom_{\mathcal{C}}(A, B))$
- (n+1)-cells: $\hom_{s\mathcal{S}et}(\Delta^n, \hom_{\mathcal{C}}(A, B))$

For a metric space ($\mathbb{R}$-category) $\mathcal{M}$, we want to know about

- which points are within distance d of each other: $\hom_{\mathbb{R}}(d, \hom_{\mathcal{M}}(A, B))$

Fuzzy sets

Given a thin monoidal category $\mathbb{W}$ (i.e., a poset with unital, associative multiplication that preserves the order), let $\mathcal{S}et(\mathbb{W})$ denote the category of $\mathbb{W}$-*fuzzy sets* with

- objects: pairs (S, f) of a set S and a function $f : S \to \mathbb{W}$
- morphisms $(S, f) \to (T, g)$: functions $t : S \to T$ such that:

$$\begin{array}{ccc} S & \xrightarrow{t} & T \\ & \searrow^{f} & \downarrow{g} \\ & & \mathbb{W} \end{array}$$

Given a fuzzy set (S, f) and $w \in \mathbb{W}$, can consider $\hom_{\mathcal{S}et(\mathbb{W})}(w, (S, f))$, the set of elements of 'weight' more than w.

Relative monoidal categories

In general, when working with enriched categories, we often want to extract certain 'shapes' from the hom-objects.

Definition

For a monoidal category $(\mathcal{W}, 1, \cdot)$, say that a monoidal category $(\mathcal{V}, I, \otimes)$ with a strong monoidal functor $\iota : \mathcal{W} \to \mathcal{V}$ is a $\mathcal{W}$-*relative monoidal category*.
This induces $よ_\iota : \mathcal{V} \to \widehat{\mathcal{W}}$ where $よ_\iota(V)(W) := \hom_{\mathcal{V}}(\iota W, V)$.
This induces a change-of-base functor $よ_\iota^* : \mathcal{V}\text{-}\mathcal{C}at \to \widehat{\mathcal{W}}\text{-}\mathcal{C}at$.

Example

Given $(\mathcal{W}, 1, \cdot)$, Day convolution puts a monoidal structure on $\widehat{\mathcal{W}}$ for which the Yoneda embedding is strong monoidal.

Interpretation

Given a $\mathcal{W}$-relative monoidal category $\mathcal{V}$:

- We give an interpretation in any $\mathcal{V}$-category $\mathcal{C}$ by interpreting the syntax in the $\widehat{\mathcal{W}}$-category $よ_\iota^*\mathcal{V}$.
- When $\mathcal{W}$ is $*$ and $\mathcal{V}$ is $\mathcal{S}et$, we recover the classical story.
- Sound and complete with respect to all $\mathcal{W}$-relative monoidal categories $\mathcal{V}$ and all $\mathcal{V}$-categories[2].
- Also sound and complete with respect to all all $\widehat{\mathcal{W}}$-categories[2].
- Since the restricted Yoneda embedding is always fully faithful, we can try to identify its image and add rules to the syntax that restrict the interpretation to this image, so that the syntax is complete with respect to $\mathcal{V}$-categories[2].
- In the paper, we do this when $\mathcal{W}$ is thin and $\mathcal{V}$ is the category of $\mathcal{W}$-fuzzy sets[2].

[2] with some universal properties

Judgments for enriched simply typed lambda calculus

Start with a (thin) monoidal category $\mathbb{W}$. Judgments:

- Γ ctx (list of types e.g., $x : R, y : S, z : T$)
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Judgments for enriched simply typed lambda calculus

Start with a (thin) monoidal category $\mathbb{W}$. Judgments:

- w wt
- $w = w'$ wt
- $w \leqslant w'$ wt
- Γ ctx (list of types e.g., $x : R, y : S, z : T$)
- T type
- $\Gamma \vdash_w t : T$
- $S = T$ type
- $\Gamma \vdash_w s = t : T$

Structural rules

$$\frac{\Gamma \vdash t :_w T \qquad v \leqslant w \text{ wt}}{\Gamma \vdash t :_v T}\ \mathbb{W}\text{-}\textsc{Wk}$$

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$$\dfrac{\Gamma, x : T, \Delta \text{ ctx}}{\Gamma, x : T, \Delta \vdash x :_1 T}\ \textsc{Var} \qquad \dfrac{\Gamma, \Delta \vdash t :_w T \qquad S \text{ type}}{\Gamma, s : S, \Delta \vdash t :_w T}\ \textsc{Wk}$$

$$\dfrac{\Gamma, x : S, \Delta \vdash t :_w T \qquad \Gamma \vdash s :_v S}{\Gamma, \Delta \vdash t[s/x] :_{vw} T}\ \textsc{Subst}$$

Type formers

- Rules for unary weighted products:

$$\textsc{Wtg-Form}\quad \frac{T \text{ type} \qquad w \text{ wt}}{T^w \text{ type}}$$

$$\textsc{Wtg-Intro}\quad \frac{\Gamma \vdash t :_{vw} T \qquad w \text{ wt}}{\Gamma \vdash t^w :_v T^w}$$

$$\textsc{Wtg-Elim}\quad \frac{\Gamma \vdash t :_v T^w}{\Gamma \vdash t^{\backslash w/} :_{vw} T}$$

$$\textsc{Wtg-}\beta\quad \frac{\Gamma \vdash t :_{vw} T \qquad w \text{ wt}}{\Gamma \vdash t = t^{w\backslash w/} :_{vw} T}$$

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- Rules for $S \times T$ and $S \to T$: write w under every $\vdash$ in the rules for $\times, \to$ in stλc

Sematics

Definition

We call the rules above the $\mathbb{W}$-*enriched simply typed lambda calculus*.

Theorem

Consider a $\mathcal{W}$-relative monoidal category $\mathcal{V}$ with finite products and which is powered by $\mathcal{W}$. There is an interpretation of the $\mathcal{W}$-enriched simply typed lambda calculus in any cartesian closed $\mathcal{V}$-category.
This interpretation is complete.

Completeness for fuzzy sets

Consider a thin monoidal category $\mathbb{W}$. If $\mathbb{W}$ is cartesian closed (i.e., has finite meets and implication), then $\mathcal{S}et(\mathbb{W})$ meets the hypotheses.

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Presheaves P such that:

- the restriction maps of P are monic (i.e., subset inclusions);
- for all $w \in \mathbb{W}$ and $x \in P(w)$, the subset $\{v \mid x \in P(v)\} \subseteq \mathbb{W}$ has a maximal element.

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Adding the following rules, we restrict the interpretation to this full subcategory, and we obtain completeness for $\mathcal{S}et(\mathbb{W})$-categories.

$$\frac{\Gamma \vdash s, t :_w T \qquad v \leqslant w \text{ wt} \qquad \Gamma \vdash s = t :_v T}{\Gamma \vdash s = t :_w T} \qquad \frac{\Gamma \vdash t :_w T}{\max(t) \text{ wt}}$$

$$\frac{\Gamma \vdash t :_w T}{w \leqslant \max(t) \text{ wt}} \qquad \frac{\Gamma \vdash t :_w T}{\Gamma \vdash t :_{\max(t)} T}$$

Outline

Enriched natural deduction

If we erase the terms from our enriched simply typed lambda calculus, we obtain an enriched natural deduction with the Curry-Howard correspondence baked in.

$$\textsc{Var}\ \frac{\Gamma, T, \Delta\ \text{ctx}}{\Gamma, T, \Delta \vdash_1 T} \qquad \textsc{Wk}\ \frac{\Gamma, \Delta \vdash_w T \qquad S\ \text{type}}{\Gamma, S, \Delta \vdash_w T} \qquad \textsc{Subst}\ \frac{\Gamma, S, \Delta \vdash_w T \qquad \Gamma \vdash_v S}{\Gamma, \Delta \vdash_{vw} T}$$

$$\mathbb{W}\text{-}\textsc{Wk}\ \frac{\Gamma \vdash_w T \qquad v \leqslant w\ \text{wt}}{\Gamma \vdash_v T} \qquad \textsc{Wtg-Form}\ \frac{T\ \text{type} \qquad w\ \text{wt}}{T^w\ \text{type}} \qquad \textsc{Wtg-Intro}\ \frac{\Gamma \vdash_{vw} T \qquad w\ \text{wt}}{\Gamma \vdash_v T^w}$$

...

Fuzzy natural deduction

We then obtain a 'fuzzy' logic that we might want to compare to classical fuzzy logic.

- There, propositions take truth values not necessarily in $\{0, 1\}$, but in a poset like $[0, 1]$.
- Though there has been work on intuitionistic fuzzy logic and natural deduction, the syntaxes in the literature do not express 'fuzziness': there are only judgments like $\Gamma \vdash T$ which are interpreted as meaning that T has truth value 1 whenever the constituent types of Γ do.
- Thus, we obtain a 'truly fuzzy' natural deduction.

Summary

We have:

- An enriched simply typed lambda calculus (parametrized by a monoidal category $\mathcal{W}$)
- An enriched natural deduction (parametrized by a monoidal category $\mathcal{W}$)
- Sound and complete interpretation in $\widehat{\mathcal{W}}$-categories (analogous for enriched natural deduction)
- Sound and complete interpretation in $\mathcal{S}et(\mathcal{W})$-categories with additional rules (analogous for enriched natural deduction)
- New notion of natural deduction for fuzzy logic

Thank you!