Fuzzy and enriched type theory

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Outline

Motivation

Enriched categories

Simple type theory

Dependent type theory

Internal languages

Type theories give us internal languages for various kinds of categorical structures.

simply typed lambda calculus	$\leftrightsquigarrow$	cartesian closed categories
Martin-Löf type theory	$\leftrightsquigarrow$	lcc categories with fibrations
linear type theory	$\leftrightsquigarrow$	monoidal (*-autonomous) categories
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enriched type theory	$\leftrightsquigarrow$	enriched categories

Fuzzy logic

Propositions are not valued in the booleans, but in a more general ordered monoid, such as $[0, 1]$.

Used in

- control: Sendai subway, washing machines, camera autofocus
- medical diagnosis: CADIAG-2
- data analysis: fuzzy clustering in image segmentation
- knowledge representation: fuzzy description logics

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Example (CADIAG-2)

"IF suspicion of liver metastases by liver palpation THEN pancreatic cancer with the confirmability degree 0.3"

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Goal: a *proof relevant* fuzzy logic.

Fuzzy logic ⇜⇝ enriched categories

- Here: Fuzzy logic as the internal language of categories enriched in fuzzy sets.
- Categories = categories enriched in sets.
- Fuzzy sets = sets + fuzziness
- They provide a good first example.

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Definition

A *category* C *enriched in* a monoidal category $(\mathcal{V}, \mathbb{I}, \otimes)$ consists of

- a collection $\mathrm{ob}\, C$ of objects
- an object $C(A, B)$ of $\mathcal{V}$ for $A, B \in \mathrm{ob}\, C$
- a morphism $\mathbb{I} \to C(A, A)$ for each $A \in \mathrm{ob}\, C$
- a morphism $C(A, B) \otimes C(B, C) \to C(A, C)$ for $A, B, C \in \mathrm{ob}\, C$

such that ...

Enriched categories

Enriched categories are everywhere!

Enriched in	Get
sets	categories
abelian groups	preadditive categories
R-modules	R-linear categories
categories	2-categories
simplicial sets	simplicial categories
...	

Examples: Enriching in posets

- Enriched in the booleans $\mathbb{B} = (\{0, 1\}, \wedge, \top)$: preordered sets.

$$\begin{aligned} \hom(x, y) &\leftrightsquigarrow x \leq y \\ \top \leq \hom(x, x) &\leftrightsquigarrow x \leq x \\ \hom(x, y) \wedge \hom(y, z) \leq \hom(x, z) &\leftrightsquigarrow x \leq y \wedge y \leq z \Rightarrow x \leq z \end{aligned}$$

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- Enriched in $\mathbb{L} = ([0,\infty], +, 0)$: (Lawvere) metric spaces.

$$\begin{aligned} \hom(x,y) &\leftrightsquigarrow d(x,y) \\ 0 \geq \hom(x,x) &\leftrightsquigarrow 0 \geq d(x,x) \\ \hom(x,y) + \hom(y,z) \geq \hom(x,z) &\leftrightsquigarrow d(x,y) + d(y,z) \geq d(x,z) \end{aligned}$$

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- $\mathbb{I} = ([0,1], \cdot, 1) \cong \mathbb{L}$, the interval
- Enriched in a monoidal poset $\mathbb{W}$ (e.g., $\mathbb{B}$, $\mathbb{I}$, $\mathbb{L}$, etc.): a setting for propositional fuzzy logic.

$$\hom(x,y) \in \mathbb{W} \quad 1 \leq \hom(x,x) \quad \hom(x,y) \cdot \hom(y,z) \leq \hom(x,z)$$

Example: Enriching in fuzzy sets

Definition

The category $\mathcal{S}et(\mathbb{W})$ of $\mathbb{W}$-*fuzzy sets* has

- objects: pairs $(S, |-|_S)$ of a set S and a function $|-|_S : S \to \mathbb{W}$
- morphisms $(S, |-|_S) \to (T, |-|_T)$: functions $f : S \to T$ such that $|x|_S \leq |f(x)|_T$

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Definition

A *($\mathbb{W}$-)fuzzy category* is a category enriched in $\mathbb{W}$-fuzzy sets.

Fuzzy categories

Observation

A $\mathbb{W}$-*fuzzy category* (i.e., $\mathcal{S}et(\mathbb{W})$-enriched category) is a category together with a weight $|f| \in \mathbb{W}$ for each morphism f such that $1 \leq |\mathrm{id}_A|$ and $|f| \cdot |g| \leq |g \circ f|$.

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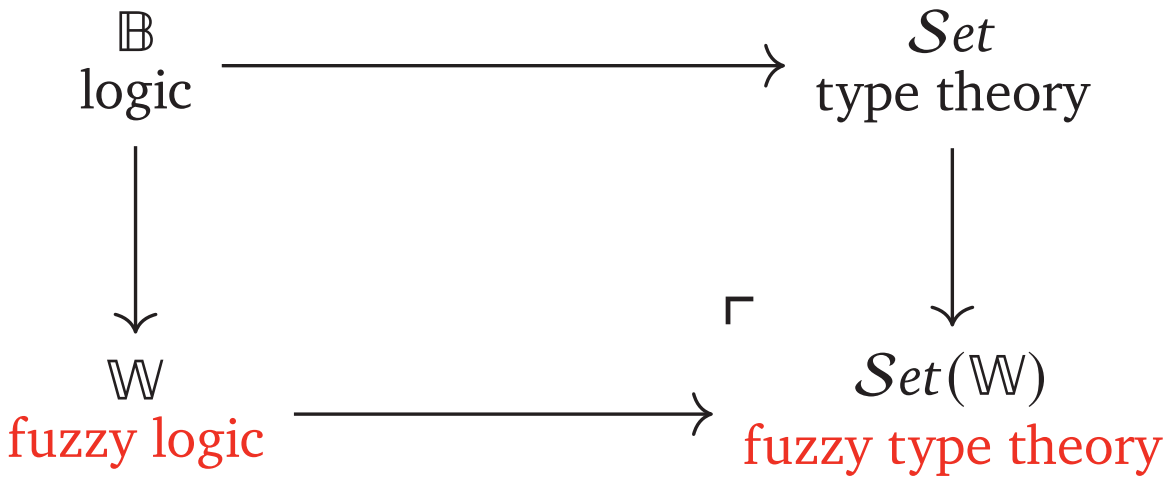
Example

If $\mathbb{W}$ is closed and has all meets, then we obtain a $\mathbb{W}$-fuzzy category $\underline{\mathcal{S}et(\mathbb{W})}$: objects are pairs $(S, |-|_S)$, morphisms are functions, $|f : S \to T|$ is given by $\bigwedge_{x \in S} \frac{|f(x)|_T}{|x|_S}$.

Fuzzy logic and type theory via enrichment



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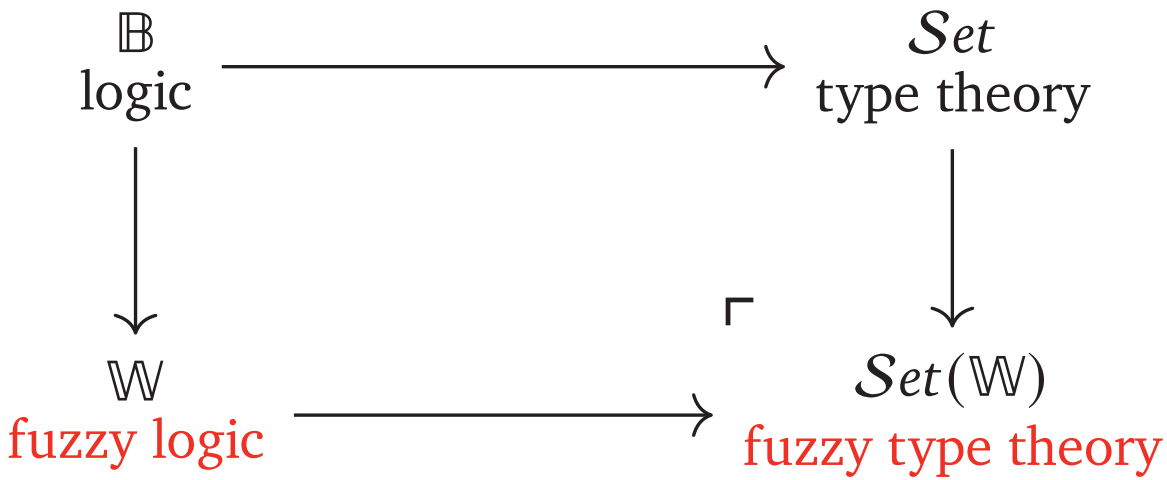

	type theory	fuzzy type theory
types	propositions	claims
terms	proofs	reasons

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Judgments and structural rules

Judgments of $\mathbb{W}$-STλC, for a monoidal poset $\mathbb{W}$
T type $\quad \Gamma \vdash t : T \quad \Gamma \vdash s = t : T$

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w wt	$v \leq w$ wt	T type	$\Gamma \vdash t :_w T$	$\Gamma \vdash s = t :_w T$

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Judgments of $\mathbb{W}$-STλC, for a monoidal poset $\mathbb{W}$

$$w\ \mathsf{wt} \qquad v \leq w\ \mathsf{wt} \qquad T\ \mathsf{type} \qquad \Gamma \vdash t :_w T \qquad \Gamma \vdash s = t :_w T$$

Structural rules of $\mathbb{W}$-STλC

$$\textsc{Var}\ \frac{\Gamma, x : T, \Delta\ \mathsf{ctx}}{\Gamma, x : T, \Delta \vdash x :_1 T} \qquad \mathbb{W}\textsc{-Wk}\ \frac{\Gamma \vdash t :_w T \qquad v \leq w\ \mathsf{wt}}{\Gamma \vdash t :_v T}$$

$$\textsc{Subst}\ \frac{\Gamma, x : S, \Delta \vdash t :_w T \qquad \Gamma \vdash s :_v S}{\Gamma, \Delta \vdash t[s/x] :_{vw} T}$$

Type formers

Type formers of $\mathbb{W}$-STλC

$$\textsc{Wtg-Intro}\quad \frac{\Gamma \vdash t :_{vw} T \qquad w\ \mathsf{wt}}{\Gamma \vdash t^w :_v T^w}$$

$$\times\textsc{-Intro}\quad \frac{\Gamma \vdash s :_w S \qquad \Gamma \vdash t :_w T}{\Gamma \vdash \langle s, t \rangle :_w S \times T}$$

$$\rightarrow\textsc{-Intro}\quad \frac{\Gamma, x : S \vdash t :_w T}{\Gamma \vdash \lambda x.t :_w S \rightarrow T}$$

$\ldots$

Approach to semantics

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- Given a fuzzy set $(S, |-|)$ and $w \in \mathbb{W}$, interested in $\{s \in S \mid |s| \geq w\}$.
- This is functorial in $(S, |-|)$, w and is the restricted Yoneda embedding $\hom(\iota -, -) : \mathcal{S}et(\mathbb{W}) \to \widehat{\mathbb{W}}$.

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- Extraction of 'shapes' is a common move in category theory.

Semantics

Theorem

Consider $\iota : \mathbb{W} \to \mathcal{V}$ strong monoidal, where $\mathcal{V}$ has finite products and is powered by $\mathbb{W}$.

There is an interpretation of $\mathbb{W}$-STλC in any cartesian closed $\mathcal{V}$-category. It is complete for $\widehat{\mathbb{W}}$-categories.

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$\mathbb{W} = *$, $\mathcal{V} = \mathcal{S}et$: the classical story.

Completeness for fuzzy sets

This is complete for $\widehat{\mathbb{W}}$ but not for $\mathcal{S}et(\mathbb{W})$.

Recall: $\hom(\iota -, -) : \mathcal{S}et(\mathbb{W}) \hookrightarrow \widehat{\mathbb{W}}$. What is its image?

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Presheaves P such that:

- the restriction maps of P are monic (i.e., subset inclusions);
- for all $x \in P(w)$, the subset $\{v \mid x \in P(v)\} \subseteq \mathbb{W}$ has a maximal element.

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Adding the following rules: completeness for $\mathcal{S}et(\mathbb{W})$-categories.

$$\frac{\Gamma \vdash s, t :_w T \qquad v \leq w \qquad \Gamma \vdash s = t :_v T}{\Gamma \vdash s = t :_w T} \qquad \frac{\Gamma \vdash t :_w T}{w \leq \max(t)\ \mathsf{wt}}$$

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Fuzzy dependent type theory

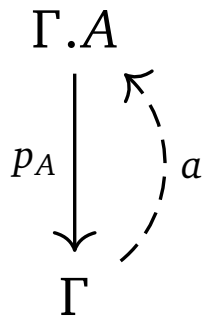
Joint with S. Arya, G. Coraglia, J. O'Connor, H. Riess, A. L. Tenorio.

Fix a monoidal poset $\mathbb{W}$ (e.g., $[0, 1]$).

Definition

A *fuzzy display map category* is a $\mathbb{W}$-fuzzy category with a class of *display maps* $p_A : \Gamma.A \to \Gamma$ of weight 1, closed under fuzzy substitution pullback. In it:

- types $\Gamma \vdash A$ type are display maps p_A
- w-terms $\Gamma \vdash a :_w A$ are sections of p_A of weight at least w



Structural rules

$$\textsc{Var}\ \frac{\vdash \Gamma, x : A, \Delta\ \mathrm{ctx}}{\Gamma, x : A, \Delta \vdash x :_1 A} \qquad \mathbb{W}\textsc{-Wk}\ \frac{\Gamma \vdash a :_w A \qquad v \leq w}{\Gamma \vdash a :_v A}$$

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$$\begin{array}{ccc} \Gamma.B[a/x] & \longrightarrow & \Gamma.A.B \\ {\scriptstyle p_{B[a/x]}}\downarrow & \lrcorner & \downarrow{\scriptstyle p_B} \\ \Gamma & \xrightarrow[a]{} & \Gamma.A \end{array}$$

Subst $\leftrightsquigarrow$ *fuzzy substitution pullbacks*.

In $\underline{\mathcal{S}et(\mathbb{W})}$, $|(\gamma, b) \in \Gamma.B[a/x]| := \frac{|b|}{|a|} \wedge |\gamma|$

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Theorem
Sound interpretation in fuzzy display map categories.

Σ-types and Π-types

Σ-Intro

$$\frac{\Gamma \vdash a :_v A \qquad \Gamma \vdash b :_w B(a)}{\Gamma \vdash (a, b) :_{vw} \Sigma_{x:A} B(x)}$$

Π-Elim

$$\frac{\Gamma \vdash a :_v A \qquad \Gamma \vdash f :_w \Pi_{x:A} B(x)}{\Gamma \vdash f(a) :_{vw} B(a)}$$

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Example (CADIAG-2), with $\mathbb{W} = ([0,1], \cdot)$

$r :_{0.3} \Pi_{p:\mathsf{Patient}} \mathsf{Susp}(p) \to \mathsf{PancCancer}(p)$

$\gamma :_1 \mathsf{Patient} \qquad \mathsf{obs} :_{0.7} \mathsf{Susp}(\gamma)$

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By Π-elimination: $\quad r(\gamma)(\mathsf{obs}) :_{0.21} \mathsf{PancCancer}(\gamma)$

Summary

We have:

- An enriched simply typed lambda calculus and natural deduction (parametrized by a monoidal poset $\mathbb{W}$)
- Sound and complete interpretation in $\widehat{\mathbb{W}}$-categories, and in $\mathcal{S}et(\mathbb{W})$-categories with additional rules
- A fuzzy dependent type theory, with a sound interpretation in $\mathcal{S}et(\mathbb{W})$
- Potential applications in domains using fuzzy logic, like medical diagnosis

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Future work:

- Completeness for the dependent theory; other $\mathcal{V}$
- Inductive types
- Normalization, implementation

Thank you!