The Univalence Principle

Paige Randall North

University of Pennsylvania

Joint work with Benedikt Ahrens, Michael Shulman, and Dimitris Tsementzis

arXiv:2004.06572

2 December 2020

Outline

Different notions of equality

Synthetic vs. analytic equalities

In MLTT, we always have a (*synthetic*) equality type between $a, b : T$

$$a =_T b.$$

Depending on the type T, we might have a type of "*analytic* equalities"

$$a \cong b.$$

A "univalence principle" for this T and this $\cong$ states that

$$(a =_T b) \to (a \cong b)$$

is an equivalence.

Different notions of equality

Synthetic vs. analytic equalities

In MLTT, we always have a (*synthetic*) equality type between $a, b : T$

$$a =_T b.$$

Depending on the type T, we might have a type of "*analytic* equalities"

$$a \cong b.$$

A "univalence principle" for this T and this $\cong$ states that

$$(a =_T b) \to (a \cong b)$$

is an equivalence.

The univalence *axiom* in type theory states that

$$S =_{\mathcal{U}} T \to S \simeq T$$

is an equivalence.

Identicals and indiscernibilites

Identity of indiscernibles

Leibniz: two things are equal when they are *indiscernible* (have the same properties).

$$(a = b) \leftarrow \left(\forall P. P(a) \leftrightarrow P(b)\right)$$

Identicals and indiscernibilites

Identity of indiscernibles

Leibniz: two things are equal when they are *indiscernible* (have the same properties).

$$(a = b) \longleftrightarrow \big(\forall P. P(a) \longleftrightarrow P(b)\big)$$

Identicals and indiscernibilites

Identity of indiscernibles

Leibniz: two things are equal when they are *indiscernible* (have the same properties).

$$(a =_T b) \leftrightarrow \left(\prod_{P:T \to \mathcal{U}} P(a) \simeq P(b) \right)$$

Identicals and indiscernibilites

Identity of indiscernibles

Leibniz: two things are equal when they are *indiscernible* (have the same properties).

$$(a =_T b) \leftrightarrow \left(\prod_{P:T \to \mathcal{U}} P(a) \simeq P(b) \right)$$

- This holds in MLTT.

Identicals and indiscernibilites

Identity of indiscernibles

Leibniz: two things are equal when they are *indiscernible* (have the same properties).

$$(a =_T b) \leftrightarrow \left(\prod_{P:T\to\mathcal{U}} P(a) \simeq P(b) \right)$$

- This holds in MLTT.
- Given a 'univalence principle' $(a =_T b) \simeq (a \cong b)$, we would find a *structure identity principle* (in the sense of Aczel):

$$(a \cong b) \to \left(\prod_{P:T\to\mathcal{U}} P(a) \simeq P(b) \right).$$

Goal

Our goal

To define a large class of (higher) *structures* and a notion of *equivalence* between them validating a univalence principle. This then automatically validates a structure identity principle.

Using ideas from:

- *First Order Logic with Dependent Sorts*, Makkai, 1995.
- *Univalent categories and the Rezk completion*, Ahrens, Kapulkin, Shulman, 2015.

Outline

1. Motivation
2. Lower structure identity principles in univalent foundations
3. First-order logic with dependent sorts (FOLDS) for lower structures
4. FOLDS categories

h-levels

We can stratify (some) types into h-levels.

0: T is *contractible* if

$$\mathsf{isContr}(T) := \Sigma_{c:T}\Pi_{y:T}\, c =_T y$$

1: T is a *proposition* if

$$\mathsf{isProp}(T) := \Pi_{x,y:T}\mathsf{isContr}(x =_T y)$$

2: T is a *set* if

$$\mathsf{isSet}(T) := \Pi_{x,y:T}\mathsf{isProp}(x =_T y)$$

3: T is a *groupoid* if

$$\mathsf{isGpd}(T) := \Pi_{x,y:T}\mathsf{isSet}(x =_T y)$$

$n+1$: T is of *h-level* $n+1$ if

$$\mathsf{ishlevel}(n+1)(T) := \Pi_{x,y:T}\mathsf{ishlevel}(n)(x =_T y)$$

Propositions

Assuming the Univalence Axiom:

$$(S =_{\mathcal{U}} T) \simeq (S \simeq T)$$

for every type S, T:

Theorem (univalence for propositions)

Given two propositions P and Q,

$$(P =_{\mathsf{Prop}} Q) \simeq (P \leftrightarrow Q).$$

Corollary (structure identity principle for propositions)

Given two propositions P and Q,

$$(P \leftrightarrow Q) \to \left(\prod_{S:\mathsf{Prop} \to \mathcal{U}} S(P) \simeq S(Q) \right).$$

Sets

Theorem (univalence for sets)

Given two sets S and T,

$$(S =_{\mathsf{Set}} T) \simeq (S \cong T).$$

Theorem ('structure identity principle' for structures sets), Coquand-Danielsson

Given terms S, T of a type $\mathscr{S}$ of sets with structure (groups, monoids, etc),

$$(S =_{\mathscr{S}} T) \simeq (S \cong T)$$

Categories

Theorem (univalence for categories), Ahrens-Kapulkin-Shulman

Given two *univalent* categories C and D,

$$(C =_{\mathsf{UCat}} D) \simeq (C \simeq D).$$

Definition

A category C is *univalent* if $(x =_{Ob(C)} y) \simeq (x \cong y)$.

Categories

Theorem (univalence for categories), Ahrens-Kapulkin-Shulman

Given two *univalent* categories C and D,

$$(C =_{\mathsf{UCat}} D) \simeq (C \simeq D).$$

Definition

A category C is *univalent* if $(x =_{Ob(C)} y) \simeq (x \cong y)$.

Goal:

- Systematically generalize the property '*univalent*' and the notion of equivalence so that we get univalence theorems for *univalent* higher categories, *univalent* categories with structure, etc...

Magmas

Magmas

A *magma* is a set M and a binary operation $M \times M \to M$.

There are two notions of 'sameness' for elements m, n of a magma:

1. Equality: $m =_M n$
2. Indiscernibility:
   $\prod_{x,y:M}(mx = nx) \times (xm = xn) \times \big((xy = m) \leftrightarrow (xy = n)\big)$

This produces two notions of equivalence of magmas:

1. $M \cong_e N$ if there are morphisms $f : M \leftrightarrows N : g$ respecting the operation such that gfm is *equal* to m for all $m : M$ and likewise for fgn
2. $M \cong_i N$ if there are morphisms $f : M \leftrightarrows N : g$ respecting the operation such that gfm is *indiscernible* from m for all $m : M$ and likewise for fgn

Preorders and topological spaces

Preorders

A *preorder* is a set P and a reflexive, transitive relation $\leq : P \times P \to \mathsf{Prop}$. Two elements p, q of a preorder P are *indiscernible* if

$$\prod_{x:P} (p \leq x \leftrightarrow q \leq x) \times (x \leq p \leftrightarrow x \leq q) \times (p \leq p \leftrightarrow q \leq q)$$

or, equivalently, if $p \leq q \times q \leq p$.

Topological spaces

A *topological space* is a set T and a collection $O : (T \to \mathsf{Prop}) \to \mathsf{Prop}$ of 'open' subsets closed under union and finite intersection.
Two elements s, t of a topological space T are *indiscernible* if $U(s) \leftrightarrow U(t)$ for every open set U of T.

Motivation

Equivalences between (higher) categorical structures are up to indiscernibility.

A lower structure identity principle in UF

Theorem (univalence for magmas with $\cong_e$)

Given two magmas M, N,

$$(M =_{\mathsf{Mag}} N) \simeq (M \cong_e N).$$

- This is a special case of univalence for sets with structure (Coquand-Danielsson)
- The same holds for preorders with $\cong_e$ and for topological spaces with $\cong_e$.

Another lower structure identity principle in UF?

Univalence with $\cong_i$

Q: Can we hope for the same with $\cong_i$?

Another lower structure identity principle in UF?

Univalence with $\cong_i$

Q: Can we hope for the same with $\cong_i$?

A: No: for example, the projection $U : \mathsf{Mag} \to \mathsf{Set}$ would then take an equivalence $M \cong_i N$ to an equivalence $UM \cong_i UN$ between the underlying sets, making it an equivalence $M \cong_e N$.

For example, let $\mathbf{1}$ be the poset whose underlying set has one element, and let $\mathbf{2}$ be the poset whose underlying set has two elements a and b for which $a \leq b$ and $b \leq a$.

Another lower structure identity principle in UF?

Univalence with $\cong_i$

Q: Can we hope for the same with $\cong_i$?

A: No: for example, the projection $U : \mathsf{Mag} \to \mathsf{Set}$ would then take an equivalence $M \cong_i N$ to an equivalence $UM \cong_i UN$ between the underlying sets, making it an equivalence $M \cong_e N$.

A: Yes: if we identify equality and indiscernibility.

For example, let $\mathbf{1}$ be the poset whose underlying set has one element, and let $\mathbf{2}$ be the poset whose underlying set has two elements a and b for which $a \leq b$ and $b \leq a$.

Outline

First-order logic with dependent sorts

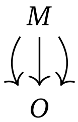
Inverse category

An *inverse category* is a strict category $\mathscr{I}$ and a function $\rho : \mathscr{I} \to \mathsf{Nat}^{\mathsf{op}}$ whose fibers are discrete.

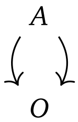
The *height* of an inverse category $(\mathscr{I}, \rho)$ is the maximum value of ρ.

Signatures

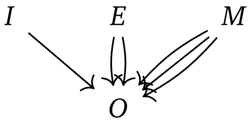
Signatures are inverse categories of finite height.



$\mathscr{L}_{\mathsf{Magma}}$



$\mathscr{L}_{\mathsf{Proset}}$



$\mathscr{L}_{\mathsf{Group}}$

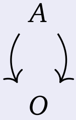
Structures

An $\mathscr{L}$-structure is a Reedy-fibrant functor from $\mathscr{L}$ into $\mathscr{U}$.

$\mathscr{L}_{\mathsf{Proset}}$-structures

An $\mathscr{L}_{\mathsf{Proset}}$-structure S is

1. A type SO,
2. A type $SA(x,y)$ for every $x, y : O$ (meaning $x \leq y$)



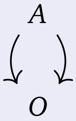
Structures

An $\mathscr{L}$-structure is a Reedy-fibrant functor from $\mathscr{L}$ into $\mathscr{U}$.

$\mathscr{L}_{\mathsf{Proset}}$-structures

An $\mathscr{L}_{\mathsf{Proset}}$-structure S is

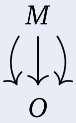
1. A type SO,
2. A type $SA(x,y)$ for every $x,y : O$ (meaning $x \leq y$)



$\mathscr{L}_{\mathsf{Magma}}$-structures

An $\mathscr{L}_{\mathsf{Magma}}$-structure S is

1. A type SO,
2. A type $SM(x,y,z)$ for every $x,y,z : O$ (meaning z is the product of x and y)



We can impose axioms on these structures.

Indiscernibilities

Indiscernibilities between O-elements of $\mathscr{L}_{\mathsf{Proset}}$-structures

An indiscernibility between two terms $p, q : SO$ consists of

- $\prod_{x:SO} SA(p,x) \cong SA(q,x)$
- $\prod_{x:SO} SA(x,p) \cong SA(x,q)$
- $SA(p,p) \cong SA(q,q)$

Indiscernibilities

Indiscernibilities between O-elements of $\mathscr{L}_{\mathsf{Proset}}$-structures

An indiscernibility between two terms $p, q : SO$ consists of

- $\prod_{x:SO} SA(p,x) \cong SA(q,x)$
- $\prod_{x:SO} SA(x,p) \cong SA(x,q)$
- $SA(p,p) \cong SA(q,q)$

Indiscernibilities between O-elements of $\mathscr{L}_{\mathsf{Magma}}$-structures

An indiscernibility between two terms $m, n : SO$ consists of

- $\prod_{x,y:SO} SM(m,x,y) \cong SM(n,x,y)$
- $\prod_{x,y:SO} SM(x,m,y) \cong SM(x,n,y)$
- $\prod_{x,y:SO} SM(x,y,m) \cong SM(x,y,n)$
- $\prod_{x:SO} SM(x,m,m) \cong SM(x,n,n)$
- $\prod_{x:SO} SM(m,x,m) \cong SM(n,x,n)$
- $\prod_{x:SO} SM(m,m,x) \cong SM(n,n,x)$
- $SM(m,m,m) \cong SM(n,n,n)$

Indiscernibilities at the top-level

Indiscernibilities between A-elements of $\mathscr{L}_{\mathsf{Proset}}$-structures

An indiscernibility between two terms $a, b : SA(p, q)$ consists of

- -

so all terms of $a, b : SA(p, q)$ are (trivially) indiscernible.

Definition (univalent structure)

A structure M of a signature $\mathscr{L}$ is *univalent* if the type of indiscernibilities between any two terms of any one sort is equivalent to the type of equalities between them.

In particular, this means that all of the top-level sorts are propositions, and all of the next-level sorts are sets.

Univalent structures

Proposition

A $\mathscr{L}_{\mathsf{Proset}}$-structure S is univalent when each $p \leq q$ is a proposition and $(p = q) \to (p \leq q) \times (q \leq p)$ is an equivalence - in other words, when A is a poset.

Proposition

A $\mathscr{L}_{\mathsf{Magma}}$-structure S is univalent when each $SM(m,n,p)$ is a proposition and $(m = n) \to \prod_{x,y:M}(mx = nx) \times (xm = xn) \times \big((xy = m) \leftrightarrow (xy = n)\big)$ is an equivalence.

Proposition

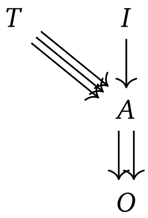
A topological space T is univalent when $(x = y) \to \prod_{U \text{ open in } T}(x \in U \leftrightarrow y \in U)$ is an equivalence – in other words, T is a T_0 space.

Outline

$\mathscr{L}_{\text{cat}}$-structures

We can define the data of a category $\mathscr{C}$ to be

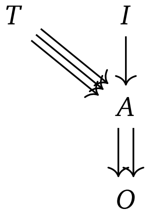
- A type $\mathscr{C}O : \mathscr{U}$
- A family $\mathscr{C}A : \mathscr{C}O \times \mathscr{C}O \to \mathscr{U}$
- A family $\mathscr{C}I : \prod_{(x:\mathscr{C}O)} \mathscr{C}A(x,x) \to \mathscr{U}$
- A family $\mathscr{C}T : \prod_{(x,y,z:\mathscr{C}O)} \mathscr{C}A(x,y) \to \mathscr{C}A(y,z) \to \mathscr{C}A(x,z) \to \mathscr{U}$



$\mathscr{L}_{\text{cat}}$-structures

We can define the data of a category $\mathscr{C}$ to be

- A type $\mathscr{C}O : \mathscr{U}$
- A family $\mathscr{C}A : \mathscr{C}O \times \mathscr{C}O \to \mathscr{U}$
- A family $\mathscr{C}I : \prod_{(x:\mathscr{C}O)} \mathscr{C}A(x,x) \to \mathscr{U}$
- A family $\mathscr{C}T : \prod_{(x,y,z:\mathscr{C}O)} \mathscr{C}A(x,y) \to \mathscr{C}A(y,z) \to \mathscr{C}A(x,z) \to \mathscr{U}$



We want to add axioms such as

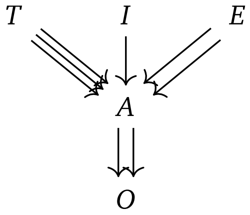
$$\begin{aligned} &\forall(x,y,z:O).\forall(f:A(x,y)).\forall(g:A(y,z)).\forall(h,h':A(x,z)). \\ &\quad T(x,y,z,f,g,h) \to T(x,y,z,f,g,h') \to (h = h') \end{aligned}$$

(composites are unique), so we add an equality 'predicate'.

$\mathscr{L}_{\text{cat}}$-structures

We can define the data of a category $\mathscr{C}$ to be

- A type $\mathscr{C}O : \mathscr{U}$
- A family $\mathscr{C}A : \mathscr{C}O \times \mathscr{C}O \to \mathscr{U}$
- A family $\mathscr{C}I : \prod_{(x:\mathscr{C}O)} \mathscr{C}A(x,x) \to \mathscr{U}$
- A family $\mathscr{C}T : \prod_{(x,y,z:\mathscr{C}O)} \mathscr{C}A(x,y) \to \mathscr{C}A(y,z) \to \mathscr{C}A(x,z) \to \mathscr{U}$



We want to add axioms such as

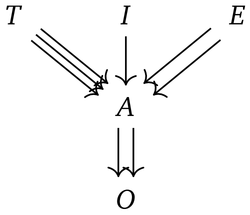
$$\forall(x,y,z : O).\forall(f : A(x,y)).\forall(g : A(y,z)).\forall(h,h' : A(x,z)).$$
$$T(x,y,z,f,g,h) \to T(x,y,z,f,g,h') \to (h = h')$$

(composites are unique), so we add an equality 'predicate'.

$\mathcal{L}_{\text{cat}}$-structures

We can define the data of a category $\mathscr{C}$ to be

- A type $\mathscr{C}O : \mathscr{U}$
- A family $\mathscr{C}A : \mathscr{C}O \times \mathscr{C}O \to \mathscr{U}$
- A family $\mathscr{C}I : \prod_{(x:\mathscr{C}O)} \mathscr{C}A(x,x) \to \mathscr{U}$
- A family $\mathscr{C}T : \prod_{(x,y,z:\mathscr{C}O)} \mathscr{C}A(x,y) \to \mathscr{C}A(y,z) \to \mathscr{C}A(x,z) \to \mathscr{U}$
- A family $\mathscr{C}E : \prod_{(x,y:\mathscr{C}O)} \mathscr{C}A(x,y) \to \mathscr{C}A(x,y) \to \mathscr{U}$



We want to add axioms such as

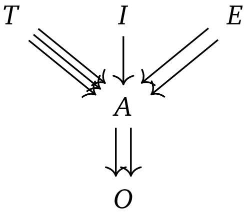
$$\forall(x,y,z:O).\forall(f:A(x,y)).\forall(g:A(y,z)).\forall(h,h':A(x,z)).$$
$$T(x,y,z,f,g,h) \to T(x,y,z,f,g,h') \to (h = h')$$

(composites are unique), so we add an equality ‘predicate’.

$\mathscr{L}_{\text{cat}}$-structures

We can define the data of a category $\mathscr{C}$ to be

- A type $\mathscr{C}O : \mathscr{U}$
- A family $\mathscr{C}A : \mathscr{C}O \times \mathscr{C}O \to \mathscr{U}$
- A family $\mathscr{C}I : \prod_{(x:\mathscr{C}O)} \mathscr{C}A(x,x) \to \mathscr{U}$
- A family $\mathscr{C}T : \prod_{(x,y,z:\mathscr{C}O)} \mathscr{C}A(x,y) \to \mathscr{C}A(y,z) \to \mathscr{C}A(x,z) \to \mathscr{U}$
- A family $\mathscr{C}E : \prod_{(x,y:\mathscr{C}O)} \mathscr{C}A(x,y) \to \mathscr{C}A(x,y) \to \mathscr{U}$



We want to add axioms such as

$$\forall(x,y,z : O).\forall(f : A(x,y)).\forall(g : A(y,z)).\forall(h,h' : A(x,z)).\\ T(x,y,z,f,g,h) \to T(x,y,z,f,g,h') \to E(h,h')$$

(composites are unique), so we add an equality ‘predicate’.

Univalent $\mathscr{L}_{\text{cat}}$-structures

- Every two elements of $\mathscr{C}I_x(f)$, $\mathscr{C}E_{x,y}(f,g)$, or $\mathscr{C}T_{x,y,z}(f,g,h)$ are indiscernible
  - so each of these types should be a proposition.
- The axioms making E a congruence for T and I make $\mathscr{C}E(f,g)$ the type of indisceribilities between $f,g : \mathscr{C}A(x,y)$
  - so we should have $(f = g) = \mathscr{C}E(f,g)$, making each $\mathscr{C}A(x,y)$ a set.
- The indiscernibilities between $a,b : \mathscr{C}O$ consist of
  1. $\phi_{x\bullet} : \mathscr{C}A(x,a) \simeq \mathscr{C}A(x,b)$ for each $x : \mathscr{C}O$
  2. $\phi_{\bullet z} : \mathscr{C}A(a,z) \simeq \mathscr{C}A(b,z)$ for each $z : \mathscr{C}O$
  3. $\phi_{\bullet\bullet} : \mathscr{C}A(a,a) \simeq \mathscr{C}A(b,b)$
  4. The following for all appropriate w,x,y,z,f,g,h:

$$T_{x,y,a}(f,g,h) \leftrightarrow T_{x,y,b}(f,\phi_{y\bullet}(g),\phi_{x\bullet}(h))$$
$$T_{x,a,z}(f,g,h) \leftrightarrow T_{x,b,z}(\phi_{x\bullet}(f),\phi_{\bullet z}(g),h)$$
$$T_{a,z,w}(f,g,h) \leftrightarrow T_{b,z,w}(\phi_{\bullet z}(f),g,\phi_{\bullet w}(h))$$
$$T_{x,a,a}(f,g,h) \leftrightarrow T_{x,b,b}(\phi_{x\bullet}(f),\phi_{\bullet\bullet}(g),\phi_{x\bullet}(h))$$
$$T_{a,x,a}(f,g,h) \leftrightarrow T_{b,x,b}(\phi_{\bullet x}(f),\phi_{x\bullet}(g),\phi_{\bullet\bullet}(h))$$
$$T_{a,a,x}(f,g,h) \leftrightarrow T_{b,b,x}(\phi_{\bullet\bullet}(f),\phi_{\bullet x}(g),\phi_{\bullet x}(h))$$
$$T_{a,a,a}(f,g,h) \leftrightarrow T_{b,b,b}(\phi_{\bullet\bullet}(f),\phi_{\bullet\bullet}(g),\phi_{\bullet\bullet}(h))$$

$$I_{a,a}(f) \leftrightarrow I_{b,b}(\phi_{\bullet\bullet}(f))$$
$$E_{x,a}(f,g) \leftrightarrow E_{x,b}(\phi_{x\bullet}(f),\phi_{x\bullet}(g))$$
$$E_{a,x}(f,g) \leftrightarrow E_{b,x}(\phi_{\bullet x}(f),\phi_{\bullet x}(g))$$
$$E_{a,a}(f,g) \leftrightarrow E_{b,b}(\phi_{\bullet\bullet}(f),\phi_{\bullet\bullet}(g))$$

Univalent $\mathscr{L}_{\text{cat}}$-structures continued

Proposition

The type of indiscernibilities between $a, b : \mathscr{C}O$ is equivalent to $a \cong b$.

Proof.

The isomorphisms $\phi_{x\bullet} : \mathscr{C}A(x,a) \cong \mathscr{C}A(x,b)$ are natural by

$$\mathscr{C}T_{x,y,a}(f,g,h) \leftrightarrow \mathscr{C}T_{x,y,b}(f,\phi_{y\bullet}(g),\phi_{x\bullet}(h))$$

(saying $\phi_{y\bullet}(g) \circ f = \phi_{x\bullet}(g \circ f)$). The rest of the data is redundent.

Thus, in a univalent $\mathscr{L}_{\text{cat}}$-structure, $(a = b) \simeq (a \cong b)$.

Theorem

Univalent $\mathscr{L}_{cat}$-structures are equivalent to the univalent categories of Ahrens-Kapulkin-Shulman.

Categorical equivalences

Theorem (univalence for univalent categories) (Ahrens-Kapulkin-Shulman)

Given univalent categories $\mathscr{C}, \mathscr{D}$,

$$(\mathscr{C} = \mathscr{D}) \simeq (\mathscr{C} \simeq \mathscr{D})$$

A categorial equivalence arises as a very surjective morphism.

A *very surjective morphism* or *equivalence* $F : \mathscr{C} \simeq \mathscr{D}$ of $\mathscr{L}_{\mathrm{cat+E}}$-structures consists of surjections

- $FO : \mathscr{C}O \twoheadrightarrow \mathscr{D}O$
- $FA : \mathscr{C}A(x,y) \twoheadrightarrow \mathscr{D}A(Fx, Fy)$ for every $x, y : \mathscr{C}O$
- $FT : \mathscr{C}T(f,g,h) \twoheadrightarrow \mathscr{D}T(Ff, Fg, Fh)$ for all $f : \mathscr{C}A(x,y), g : \mathscr{C}A(y,z), h : \mathscr{C}A(x,z)$
- $FE : \mathscr{C}E(f,g) \twoheadrightarrow \mathscr{D}E(Ff, Fg)$ for all $f, g : \mathscr{C}A(x,y)$
- $FI : \mathscr{C}I(f) \twoheadrightarrow \mathscr{D}I(Ff)$ for all $f : \mathscr{C}A(x,x)$

Categorical equivalences

Theorem (univalence for univalent categories) (Ahrens-Kapulkin-Shulman)

Given univalent categories $\mathscr{C}, \mathscr{D}$,

$$(\mathscr{C} = \mathscr{D}) \simeq (\mathscr{C} \simeq \mathscr{D})$$

A categorial equivalence arises as a very surjective morphism.

A *very surjective morphism* or *equivalence* $F : \mathscr{C} \simeq \mathscr{D}$ of univalent $\mathscr{L}_{\text{cat+E}}$-structures consists of surjections

- $FO : \mathscr{C}O \twoheadrightarrow \mathscr{D}O$
- $FA : \mathscr{C}A(x,y) \twoheadrightarrow \mathscr{D}A(Fx,Fy)$ for every $x,y : \mathscr{C}O$
- $FT : \mathscr{C}T(f,g,h) \twoheadrightarrow \mathscr{D}T(Ff,Fg,Fh)$ for all $f : \mathscr{C}A(x,y), g : \mathscr{C}A(y,z), h : \mathscr{C}A(x,z)$
- $FE : \mathscr{C}E(f,g) \twoheadrightarrow \mathscr{D}E(Ff,Fg)$ for all $f,g : \mathscr{C}A(x,y)$
- $FI : \mathscr{C}I(f) \twoheadrightarrow \mathscr{D}I(Ff)$ for all $f : \mathscr{C}A(x,x)$

Categorical equivalences

Theorem (univalence for univalent categories) (Ahrens-Kapulkin-Shulman)

Given univalent categories $\mathscr{C}, \mathscr{D}$,

$$(\mathscr{C} = \mathscr{D}) \simeq (\mathscr{C} \simeq \mathscr{D})$$

A categorial equivalence arises as a very surjective morphism.

A *very surjective morphism* or *equivalence* $F : \mathscr{C} \simeq \mathscr{D}$ of univalent $\mathscr{L}_{\mathrm{cat+E}}$-structures consists of surjections

- $FO : \mathscr{C}O \twoheadrightarrow \mathscr{D}O$
- $FA : \mathscr{C}A(x,y) \twoheadrightarrow \mathscr{D}A(Fx,Fy)$ for every $x,y : \mathscr{C}O$
- $FT : \mathscr{C}T(f,g,h) \longleftrightarrow \mathscr{D}T(Ff,Fg,Fh)$ for all $f : \mathscr{C}A(x,y), g : \mathscr{C}A(y,z), h : \mathscr{C}A(x,z)$
- $FE : \mathscr{C}E(f,g) \longleftrightarrow \mathscr{D}E(Ff,Fg)$ for all $f,g : \mathscr{C}A(x,y)$
- $FI : \mathscr{C}I(f) \longleftrightarrow \mathscr{D}I(Ff)$ for all $f : \mathscr{C}A(x,x)$

Categorical equivalences

Theorem (univalence for univalent categories) (Ahrens-Kapulkin-Shulman)

Given univalent categories $\mathscr{C}, \mathscr{D}$,

$$(\mathscr{C} = \mathscr{D}) \simeq (\mathscr{C} \simeq \mathscr{D})$$

A categorial equivalence arises as a very surjective morphism.

A *very surjective morphism* or *equivalence* $F : \mathscr{C} \simeq \mathscr{D}$ of univalent $\mathscr{L}_{\mathrm{cat+E}}$-structures consists of surjections

- $FO : \mathscr{C}O \twoheadrightarrow \mathscr{D}O$
- $FA : \mathscr{C}A(x,y) \twoheadrightarrow \mathscr{D}A(Fx, Fy)$ for every $x, y : \mathscr{C}O$
- $FT : \mathscr{C}T(f,g,h) \longleftrightarrow \mathscr{D}T(Ff, Fg, Fh)$ for all $f : \mathscr{C}A(x,y), g : \mathscr{C}A(y,z), h : \mathscr{C}A(x,z)$
- $FE : (f = g) \longleftrightarrow (Ff = Fg)$ for all $f, g : \mathscr{C}A(x,y)$
- $FI : \mathscr{C}I(f) \longleftrightarrow \mathscr{D}I(Ff)$ for all $f : \mathscr{C}A(x,x)$

Categorical equivalences

Theorem (univalence for univalent categories) (Ahrens-Kapulkin-Shulman)

Given univalent categories $\mathscr{C}, \mathscr{D}$,

$$(\mathscr{C} = \mathscr{D}) \simeq (\mathscr{C} \simeq \mathscr{D})$$

A categorial equivalence arises as a very surjective morphism.

A *very surjective morphism* or *equivalence* $F : \mathscr{C} \simeq \mathscr{D}$ of univalent $\mathscr{L}_{\text{cat+E}}$-structures consists of surjections

- $FO : \mathscr{C}O \twoheadrightarrow \mathscr{D}O$
- $FA : \mathscr{C}A(x,y) \cong \mathscr{D}A(Fx, Fy)$ for every $x, y : \mathscr{C}O$
- $FT : \mathscr{C}T(f,g,h) \longleftrightarrow \mathscr{D}T(Ff, Fg, Fh)$ for all $f : \mathscr{C}A(x,y), g : \mathscr{C}A(y,z), h : \mathscr{C}A(x,z)$
- $FE : (f = g) \longleftrightarrow (Ff = Fg)$ for all $f, g : \mathscr{C}A(x,y)$
- $FI : \mathscr{C}I(f) \longleftrightarrow \mathscr{D}I(Ff)$ for all $f : \mathscr{C}A(x,x)$

Equivalences in general

Definition (equivalence)

An *equivalence* $M \simeq N$ between two $\mathscr{L}$-structures is a very split-surjective morphism $M \to N$.

Theorem

Given two univalent $\mathscr{L}$-structures M and N,

$$(M = N) \simeq (M \simeq N).$$

Theorem

For a signature $L : \mathsf{Sig}(n)$, the type of univalent L-structures is of h-level $n+1$.

Example: magmas

Equivalences of univalent magmas

An equivalence of magmas N, P consists of surjections

- $FO : NO \twoheadrightarrow PO$
- $FM : NM(x,y,z) \twoheadrightarrow PM(Fx,Fy,Fz)$

Example: magmas

Equivalences of univalent magmas

An equivalence of univalent magmas N, P consists of surjections

- $FO : NO \twoheadrightarrow PO$
- $FM : NM(x, y, z) \twoheadrightarrow PM(Fx, Fy, Fz)$

Example: magmas

Equivalences of univalent magmas

An equivalence of univalent magmas N, P consists of surjections

- $FO : NO \twoheadrightarrow PO$
- $FM : NM(x, y, z) \longleftrightarrow PM(Fx, Fy, Fz)$

Example: magmas

Equivalences of univalent magmas

An equivalence of univalent magmas N, P consists of surjections

- $FO : NO \cong PO$
- $FM : NM(x,y,z) \longleftrightarrow PM(Fx, Fy, Fz)$

Summary

For every signature $\mathscr{L}$, we have

- a notion of indiscernibility within each sort,
- a notion of univalent structures,
- a notion of equivalence,
- a univalence theorem,
- and thus a (higher) structure identity principle.

Summary

For every signature $\mathcal{L}$, we have

- a notion of indiscernibility within each sort,
- a notion of univalent structures,
- a notion of equivalence,
- a univalence theorem,
- and thus a (higher) structure identity principle.

The paper includes examples of

- †-categories,
- presheaves,
- profunctors,
- semi-displayed categories,
- bicategories,
- ...

Further work

- Drop the splitness condition for certain structures
- Formulate an analogue to the Rezk completion

Thank you!