A categorical perspective on opinion dynamics

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- Logic of opinions
  - Model with *fuzzy* lattices and *fuzzy* up-sets.
  - Above, we answer "Is $P \leq Q$?" or "Does P hold?" with "yes" or "no", i.e., "0" or "1".
  - Now we answer "Is $P \leq Q$?" or "Does P hold?" with a value in an ordered monoid, for instance $[0, 1]$.

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- We gain many computational tools and constructions from the existing, extensive development of category theory.

Weighted meets and joins

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- $(B\,^1\!\wedge^{\alpha} S \leq C) = \alpha$ and $(B\,^1\!\wedge^{\alpha} S \leq {}^{\alpha}C) = 1$

Back to fuzzy concepts

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- Then the set of functions $S \to [0, 1]$ has the structure of a fuzzy lattice.
- (This is the presheaf lattice $\widehat{S}$.)
- It is the *completion* of S under weighted meets and joins.
- The elements are of the form

$$P^{\alpha} \wedge^{\beta} Q \wedge^{\gamma} R \quad \text{or} \quad ((P, \alpha), (Q, \beta), (R, \gamma))$$

for $\alpha, \beta, \gamma \in [0, 1]$.

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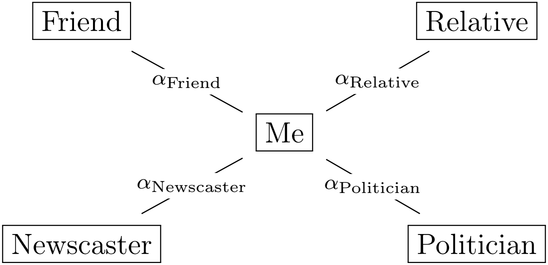
for $\alpha, \beta, \gamma \in [0,1]$ such that $\beta \cdot \rho \leq \gamma$.

Fuzzy Laplacian



- Each person x
  - has their own fuzzy lattice L_x of opinions, and
  - communicates that they hold a proposition $P_x \in L_x$.
- Then we can consider the *α-fuzzy Laplacian* at Me:

$$L(P)_{Me} := \bigwedge_{\text{people } x}^{\alpha_x} F^R_{\text{Me},x} F^L_{x,\text{Me}}(P_x)$$

Fuzzy global sections

- Instead of global sections
  - (a collection P_x such that
    $$F^L_{x,y} P_x = F^L_{y,x} P_y$$
    for all people x, y)
- we consider *fuzzy* global sections:
  - given a function $\beta : \{\text{people}\}^2 \to [0,1]$, a collection P_x such that
    $$\beta(x,y) \leq (F^L_{x,y} P_x \leq F^L_{y,x} P_y)$$
    for all people x, y.

Theorem (GNR)

The fuzzy lattice of β-fuzzy global sections is collection of fixed points of $\mathsf{id} \wedge L^\beta$ where L^β is the β-fuzzy Laplacian.

Generalization

- In addition to supplying well-behaved algebraic operations (weighted meets and joins, presheaves), the categorical perspective has another advantage.
- It easily admits generalization.
- We have thought of categories enriched in $[0, 1]$.
- We could think of categories enriched in any ordered monoid.
  - For example, for a fixed set S:

$$S \to [0, 1]$$

- We expect to be able to generalize to categories enriched in other categories.
  - For example:

$$\sum_{S \in \mathcal{S}et} S \to [0, 1]$$