Directed weak factorization systems and type theories

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Outline

Introduction: weak factorization systems and type theory

Sharpening the connection between weak factorization systems and type theory

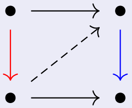
Directed homotopy type theory

Weak factorization systems

Definition of *weak factorization system*

Let $\mathcal{C}$ be a category. A weak factorization system consists of subclasses $\mathcal{L}, \mathcal{R} \subseteq \text{morphisms}(\mathcal{C})$ such that

1. every morphism $f : X \to Y$ of $\mathcal{C}$ has a factorization $X \xrightarrow{\lambda_f} Mf \xrightarrow{\rho_f} Y$ into $\mathcal{L}, \mathcal{R}$
2. every morphism of $\mathcal{L}$ lifts against every morphism of $\mathcal{R}$ (written $\mathcal{L} \boxslash \mathcal{R}$)



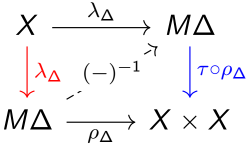
3. $\mathcal{L}$ is exactly the class of morphisms that lift on the left against all morphisms in $\mathcal{R}$ (written $\mathcal{L} = {}^{\boxslash}\mathcal{R}$)
4. $\mathcal{R}$ is exactly the class of morphisms that lift on the right against all morphisms in $\mathcal{L}$ (written $\mathcal{R} = \mathcal{L}^{\boxslash}$).

Weak factorization systems and type theory

- Put two wfs together in the right way, and you get a model structure. These underlie much of abstract homotopy theory.
- Roughly: in a model structure, one wfs describes cylinder objects $X \times I$ and one wfs describes path objects X^I.

What do wfs have to do with type theory?

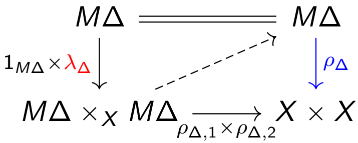
- We can factor any diagonal $X \xrightarrow{\Delta} X \times X$ into $X \xrightarrow{\lambda_\Delta} M\Delta \xrightarrow{\rho_\Delta} X \times X$.
- Then let for any points $x, y : X$, we can let $\mathsf{Id}_X(x, y) := \rho_\Delta^{-1}(x, y)$.
- For any point $x \in X$, we have $r(x) := \lambda_\Delta(x) : \mathsf{Id}_X(x, x)$.
- For any $p : \mathsf{Id}_X(x, y)$, we can construct a $p^{-1} : \mathsf{Id}_X(y, x)$.



$$\frac{\mathtt{x, y : X, p : Id_X(x, y) \vdash Id_X(y, x)} \qquad \mathtt{x : X \vdash r(x) : Id_X(x, x)}}{\mathtt{x, y : X, p : Id_x(x, y) \vdash p^{-1} : Id_X(y, x)}}$$

Weak factorization systems and type theory

- If we have $p : \mathrm{Id}_X(x, y)$ and $q : \mathrm{Id}_X(y, z)$, can we construct a $p * q : \mathrm{Id}_X(x, z)$?



$$\frac{\mathtt{x, y, z : X, p : Id_X(x, y), q : Id_X(y, z) \vdash Id_X(x, z)} \qquad \mathtt{x, y : X, p : Id_X(x, y) \vdash p : Id_X(x, y)}}{\mathtt{x, y, z : X, p : Id_x(x, y), q : Id_X(y, z) \vdash p * q : Id_X(x, z)}}$$

No: We don't know that $1_{M\Delta} \times_X \lambda_\Delta$ is in $\mathcal{L}$.

- We'll see that every model of dependent type theory with Σ and Id types induces a weak factorization system with some nice properties

Display map categories

Definition of *display map category*

Let $\mathcal{C}$ be a category with a terminal object $*$, $\mathcal{D} \subseteq \mathrm{mor}(\mathcal{C})$. $(\mathcal{C}, \mathcal{D})$ is a *display map category* if

- every morphism to $*$ is in $\mathcal{D}$,
- every isomorphism is in $\mathcal{D}$,
- pullbacks of morphisms in $\mathcal{D}$ exist
- and are in $\mathcal{D}$.

We call elements of $\mathcal{D}$ *display maps*.

- The objects of $\mathcal{C}$ represent contexts.
- $*$ represents the empty context.
- The morphisms $E \xrightarrow{p} B$ of $\mathcal{D}$ represent dependent types $b : B \vdash E(b)$ (so every context is also a type in the empty context).
- Pulling back represents substitution (so substituting into the context of a dependent type produces a new dependent type.)

Σ and Π types in display map categories

Definition of Σ types (Jacobs)

A DMC $(\mathcal{C}, \mathcal{D})$ *models* Σ *types* when $\mathcal{D}$ is closed under composition.

Definition of Π *types* (Jacobs)

A DMC $(\mathcal{C}, \mathcal{D})$ *models* Π *types* when for all

$$W \xrightarrow{g} X \xrightarrow{f} Y$$

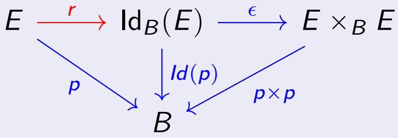
there is a display map $\Pi_f g$ representing

$$\mathrm{hom}_{\mathcal{C}/X}(f^* -, g) : (\mathcal{C}/Y)^{op} \to \mathit{Set}.$$

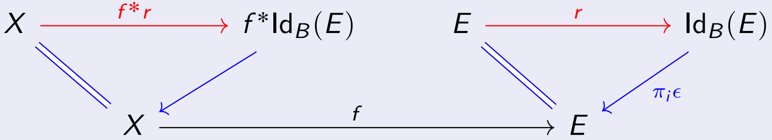
Id types

Definition of Id *types*

A DMC $(\mathcal{C}, \mathcal{D})$ with Σ types *models (Paulin-Mohring)* Id *types* when for every display map $E \xrightarrow{p} B$ of $\mathcal{C}$, there is a factorization of the diagonal



such that ϵ is in $\mathcal{D}$ and every pullback f^*r of r as shown below has the left lifting property against $\mathcal{D}$ (or: is in ${}^{\boxslash}\mathcal{D}$).



The weak factorization system

- Σ and Id types produce a factorization of any map $f : X \to Y$

$$X \xrightarrow{f^* r} X \times_Y \mathrm{Id}(Y) \xrightarrow{\pi_1 \epsilon \pi_1} Y$$

- This generates a weak factorization system $({}^{\boxslash}\mathcal{D}, \overline{\mathcal{D}})$ where $\overline{\mathcal{D}}$ is $({}^{\boxslash}\mathcal{D})^{\boxslash}$ or, equivalently, the retract closure of $\mathcal{D}$. (Gambino-Garner)
- Every model of Σ and Id lives in a weak factorization system.
- Moreover, this weak factorization system is *itself* a model.

Theorem (N)

Let $\mathcal{C}$ be a Cauchy complete category. Let $(\mathcal{C}, \mathcal{D})$ be a DMC modeling Σ and Id types. Then $(\mathcal{C}, \overline{\mathcal{D}})$ is a DMC modeling Σ and Id types.
If $(\mathcal{C}, \mathcal{D})$ also models Π types, then $(\mathcal{C}, \overline{\mathcal{D}})$ models Π types.

- So if we're given a wfs $(\mathcal{L}, \mathcal{R})$ in $\mathcal{C}$ and we want to know if it harbours a model, we only have to understand $(\mathcal{C}, \mathcal{R})$, not every $(\mathcal{C}, \mathcal{D})$ for which $\overline{\mathcal{D}} = \mathcal{R}$.

The characterization

Theorem (N)

Consider a category $\mathcal{C}$ with finite limits. The following properties of any weak factorization system $(\mathcal{L}, \mathcal{R})$ on $\mathcal{C}$ are equivalent:

1. $(\mathcal{C}, \mathcal{R})$ is a display map category modeling Σ and Id types;
2. every map to the terminal object is in $\mathcal{R}$ and $\mathcal{L}$ is stable under pullback along $\mathcal{R}$;
3. it is generated by a Moore relation system.

If this holds and $\mathcal{C}$ is locally cartesian closed, then $(\mathcal{C}, \mathcal{R})$ also models Π types.

The symmetry coming into view

Defintion of *Moore relation system*

A finitely complete category $\mathcal{C}$, an endofunctor $R : \mathcal{C} \to \mathcal{C}$ with natural transformations

$$X \underset{\epsilon_1}{\overset{\epsilon_0}{\leftleftarrows}} \xrightarrow{\ \eta\ } RX$$

(which can be called a *functorial relation*), which is

- *transitive*: $\mu_X : RX\, {}_{\epsilon_1}\!\times_{\epsilon_0} RX \to RX$ for all objects X

$$\begin{array}{ccc} RX\, {}_{\epsilon_1}\!\times_{\epsilon_0} RX & \xrightarrow{\ \mu\ } & RX \\ \epsilon_0\pi_0 \downarrow\downarrow \epsilon_1\pi_1 & & \epsilon_0 \downarrow\downarrow \epsilon_1 \\ X & = & X \end{array} \qquad \begin{array}{ccc} RX & \xrightarrow{\ 1\times\eta\ } & RX\, {}_{\epsilon_1}\!\times_{\epsilon_0} RX \\ & \searrow^{=} & \downarrow \mu \\ & & RX \end{array}$$

- *homotopical*: $\tau_f : X\, {}_{\eta f}\!\times_{\zeta} R^{\square} Y \to R(X\, {}_{f}\!\times_{\epsilon_0} RY)$ for all morphisms f...
- *symmetric*: $\nu_X : RX_{\epsilon_0} \times_{\epsilon_0} RX \to RX$ for all objects X...

The symmetry coming into view

Theorem (N)

Consider a category $\mathcal{C}$ with finite limits. The following properties of any weak factorization system $(\mathcal{L}, \mathcal{R})$ on $\mathcal{C}$ are equivalent:

1. $(\mathcal{C}, \mathcal{R})$ is a display map category modeling Σ and Id types;
2. every map to the terminal object is in $\mathcal{R}$ and $\mathcal{L}$ is stable under pullback along $\mathcal{R}$;
3. it is generated by a Moore relation system.

Corollary (N)

Let $(\mathcal{L}, \mathcal{R})$ be a wfs on a finitely complete category $\mathcal{C}$ where every map to the terminal object is in $\mathcal{R}$. Then $\mathcal{L}$ is stable under pullback along $\mathcal{R}$ if and only if $(\mathcal{L}, \mathcal{R})$ admits a symmetric functorial relation.

The symmetry coming into view

Underlying the characterization theorem is an equivalence which is a restriction of the following functors:

$$F : \mathcal{W} \leftrightarrows \mathcal{I} : G$$

- $\mathcal{W}$ is the category of wfs on $\mathcal{C}$
- $\mathcal{I}$ is category of data for identity types/functorial relations
- $F(\mathcal{L}, \mathcal{R})$ takes an object X to the factorization $X \xrightarrow{\lambda_\Delta} M\Delta \xrightarrow{\rho_\Delta} X \times X$ of its diagonal
- $G(I)$ produces a wfs from an identity type as we did earlier

For an I in $\mathcal{I}$ which at each X is

$$X \xrightarrow{r} \mathsf{Id}(X) \xrightarrow{\epsilon} X \times X$$

$FG(I)$ at each X is

$$X \xrightarrow{1 \times r\Delta} X \times_{X \times X} \mathsf{Id}(X \times X) \xrightarrow{\pi_1 \epsilon \pi_1} X \times X$$

and $I \cong FG(I)$ if and only if the I is symmetric.
On the other hand, $GF(W)$ is always a wfs, but $GF(W) \cong W$ if and only if W is symmetric.

The simplest directed weak factorization system

There are two functorial relations on $\mathcal{C}at$:

$$\mathcal{C} \to \mathcal{C}^{(\cong)} \to \mathcal{C} \times \mathcal{C}$$

$$\mathcal{C} \to \mathcal{C}^{(\rightarrow)} \to \mathcal{C} \times \mathcal{C}$$

- The first is transitive, homotopical, and symmetric, and so it is a model of the Id type.
- The second is transitive and homotopical, but not the symmetry.
- It generates a wfs (via the functor G), but not one that models the Id type.
- In particular, the morphism $\mathcal{C}^{(\rightarrow)} \to \mathcal{C} \times \mathcal{C}$ is *not* in the right class of the weak factorization system.
- But the *twisted arrow category* $\hom(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ is.

Directed type theory

Goal

To develop a directed type theory.

To formalize theorems about:

- Higher category theory
- Directed homotopy theory
  - Concurrent processes
  - Rewriting

Criteria

- Directed paths are introduced as terms of a type former, hom, to be added to Martin-Löf type theory
- Transport along terms of hom
- Independence of hom and Id

How does direction come in?

Semantically

higher groupoids

higher categories
(undirected paths $\subseteq$
directed paths)

directed spaces
(directed paths $\subseteq$
undirected paths)

Rules for hom: core and op

$$\frac{T \ \text{TYPE}}{T^{\text{core}} \ \text{TYPE}}$$

$$\frac{T \ \text{TYPE}}{T^{\text{op}} \ \text{TYPE}}$$

$$\frac{T \ \text{TYPE} \qquad t : T^{\text{core}}}{it : T}$$

$$\frac{T \ \text{TYPE} \qquad t : T^{\text{core}}}{i^{\text{op}} t : T^{\text{op}}}$$

Rules for hom: formation

hom formation

$$\frac{T \;\; \text{TYPE} \qquad s : T^{\mathrm{op}} \qquad t : T}{\hom_T(s,t) \;\; \text{TYPE}}$$

Id formation

$$\frac{T \;\; \text{TYPE} \qquad s : T \qquad t : T}{\mathrm{Id}_T(s,t) \;\; \text{TYPE}}$$

Rules for hom: introduction

hom formation

$$\frac{T \ \ \text{TYPE} \qquad t : T^{\text{core}}}{1_t : \text{hom}_T(i^{\text{op}} t, it) \ \ \text{TYPE}}$$

Id introduction

$$\frac{T \ \ \text{TYPE} \qquad t : T}{r_t : \text{Id}_T(t, t) \ \ \text{TYPE}}$$

Rules for hom: right elimination and computation

hom right elimination and computation

$$\frac{T \;\; \text{TYPE} \qquad s : T^{\text{core}}, t : T, f : \hom_T(i^{\text{op}} s, t) \vdash D(f) \;\; \text{TYPE} \qquad s : T^{\text{core}} \vdash d(s) : D(1_s)}{\begin{array}{c} s : T^{\text{core}}, t : T, f : \hom_T(i^{\text{op}} s, t) \vdash e_R(d, f) : D(f) \\ s : T^{\text{core}} \vdash e_R(d, 1_s) \equiv d(s) : D(1_s) \end{array}}$$

Id elimination and computation

$$\frac{T \;\; \text{TYPE} \qquad s : T, t : T, f : \mathsf{Id}_T(s, t) \vdash D(f) \;\; \text{TYPE} \qquad s : T \vdash d(s) : D(r_s)}{\begin{array}{c} s : T, t : T, f : \mathsf{Id}_T(s, t) \vdash j(d, f) : D(f) \\ s : T \vdash j(d, r_s) \equiv d(s) : D(r_s) \end{array}}$$

Rules for hom: left elimination and computation

hom left elimination and computation

$$\frac{T \ \ \textsc{type} \qquad s : T^{\mathrm{op}}, t : T^{\mathrm{core}}, f : \hom_T(s, it) \vdash D(f) \ \ \textsc{type} \qquad s : T^{\mathrm{core}} \vdash d(s) : D(1_s)}{\begin{array}{c} s : T^{\mathrm{op}}, t : T^{\mathrm{core}}, f : \hom_T(s, it) \vdash e_L(d, f) : D(f) \\ s : T^{\mathrm{core}} \vdash e_L(d, 1_s) \equiv d(s) : D(1_s) \end{array}}$$

Id elimination and computation

$$\frac{T \ \ \textsc{type} \qquad s : T, t : T, f : \mathsf{Id}_T(s, t) \vdash D(f) \ \ \textsc{type} \qquad s : T \vdash d(s) : D(r_s)}{\begin{array}{c} s : T, t : T, f : \mathsf{Id}_T(s, t) \vdash j(d, f) : D(f) \\ s : T \vdash j(d, r_s) \equiv d(s) : D(r_s) \end{array}}$$

Syntactic results

- Transport: for a dependent type $t : T \vdash S(t)$:

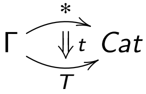
$$t : T^{\mathrm{core}}, t' : T, f : \mathrm{hom}_T(i^{\mathrm{op}}t, t'), s : S(it) \vdash \mathrm{transport}_{\mathrm{R}}(s, f) : S(t')$$

- Composition: for a type T:

$$r : T^{\mathrm{op}}, s : T^{\mathrm{core}}, t : T, f : \mathrm{hom}_T(r, is), g : \mathrm{hom}_T(i^{\mathrm{op}}s, t) \vdash \mathrm{comp}_{\mathrm{R}}(f, g) : \mathrm{hom}_T(r, t)$$

The interpretation

- Use the framework of comprehension categories
- Dependent types are represented by functors $T : \Gamma \to \mathcal{C}at$.
- Dependent terms are represented by natural transformations



  where $* : \Gamma \to \mathcal{C}at$ is the functor which takes everything to the one-object category.

- Context extension is represented by the Grothendieck construction which takes each functor $T : \Gamma \to \mathcal{C}at$ to the Grothendieck opfibration

$$\pi_\Gamma : \int_\Gamma T \to \Gamma.$$

Interpreting core and op in the empty context

$$\frac{T \text{ TYPE}}{T^{\text{core}} \text{ TYPE} \qquad T^{\text{op}} \text{ TYPE}} \qquad \frac{T \text{ TYPE} \qquad t : T^{\text{core}}}{it : T \qquad i^{\text{op}}t : T^{\text{op}}}$$

For any category T,

- $T^{\text{core}} := \text{ob}(T)$
- $T^{\text{op}} := T^{\text{op}}$
- $i : T^{\text{core}} \to T$ and $i^{\text{op}} : T^{\text{core}} \to T^{\text{op}}$ are the identity on objects.

Interpreting hom formation and introduction

$$\frac{T \ \ \text{TYPE} \qquad s : T^{\text{op}} \qquad t : T}{\hom_T(s, t) \ \ \text{TYPE}} \qquad\qquad \frac{T \ \ \text{TYPE} \qquad t : T^{\text{core}}}{1_t : \hom_T(i^{\text{op}} t, it) \ \ \text{TYPE}}$$

For any category T,

- Take the functor

$$\hom : T^{\text{op}} \times T \to \mathit{Set} \hookrightarrow \mathit{Cat}.$$

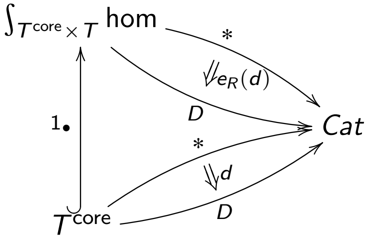
- Take the natural transformation

$$T^{\text{core}} \underset{\hom \circ (i^{\text{op}} \times i)}{\overset{*}{\rightrightarrows}} \mathit{Cat} \qquad \Downarrow 1_\bullet$$

  where each component $1_t : * \to \hom(t, t)$ picks out the identity morphism of t.

Interpreting right hom elimination and computation

$$\frac{T \ \ \text{TYPE} \qquad s : T^{\text{core}}, t : T, f : \hom_T(i^{\text{op}} s, t) \vdash D(f) \ \ \text{TYPE} \qquad s : T^{\text{core}} \vdash d(s) : D(1_s)}{\begin{array}{c} s : T^{\text{core}}, t : T, f : \hom_T(i^{\text{op}} s, t) \vdash e_R(d, f) : D(f) \\ s : T^{\text{core}} \vdash e_R(d, 1_s) \equiv d(s) : D(1_s) \end{array}}$$



- Use the fact that the subcategory T^{core} is coreflective:
  - for every $(s, t, f) \in \int_{T^{\text{core}} \times T} \hom$ there is a unique morphism $(1_s, f) : (s, s, 1_s) \to (s, t, f)$ with domain in T^{core}
- Set $e_R(d)_{(s,t,f)} := D(1_s, f) d_{(s,s,1_s)}$

Interpreting left hom elimination and computation

$$\frac{T \;\; \text{TYPE} \qquad s : T^{\mathrm{op}}, t : T^{\mathrm{core}}, f : \hom_T(s, it) \vdash D(f) \;\; \text{TYPE} \qquad s : T^{\mathrm{core}} \vdash d(s) : D(1_s)}{s : T^{\mathrm{op}}, t : T^{\mathrm{core}}, f : \hom_T(s, it) \vdash e_L(d, f) : D(f) \qquad s : T^{\mathrm{core}} \vdash e_L(d, 1_s) \equiv d(s) : D(1_s)}$$

- Replace T by T^{op} and apply right hom elimination and computation.

The homotopy theory

- The right class of the wfs generated by $\mathcal{C}^{\rightarrow}$ are those functors $E \xrightarrow{p} B$ which have the enriched right lifting property

$$\begin{array}{ccc} * & \longrightarrow & E \\ {\scriptstyle \mathrm{DOM}}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ (\rightarrow) & \longrightarrow & B \end{array}$$

- so all Grothendieck opfibrations (dependent projections) are in the right class.
- The functor $T^{\mathrm{core}} \xhookrightarrow{1_\bullet} \int_{T^{\mathrm{core}} \times T} \hom$ is the left part of the factorization of

$$i : T^{\mathrm{core}} \to T.$$

- Then the right hom elimination and computation rule arises from the weak factorization system.

$$\begin{array}{ccc} T^{\mathrm{core}} & \xrightarrow{\quad d \quad} & \int_{\int_{T^{\mathrm{core}} \times T} \hom} D \\ {\scriptstyle 1_\bullet}\big\downarrow & \overset{e_R(d)}{\nearrow} & \big\downarrow{\scriptstyle \pi} \\ \int_{T^{\mathrm{core}} \times T} \hom & = & \int_{T^{\mathrm{core}} \times T} \hom \end{array}$$

Summary & future work

Summary

We have:

- a directed type theory
- with a model in *Cat*.

Future work

We need to:

- integrate this into traditional Martin-Löf type theory
  - integrate Id and hom in the same theory
  - specify Σ, Π, etc
- find interpretations in categories of directed spaces
  - build 'directed' weak factorization systems
  - build universes

Thank you!

Further Reading

E. Finster and S. Mimram.
A Type-Theoretical Definition of Weak ω-Categories.
2017 32nd Annual ACM/IEEE Symposium on Logic in Computer Science, 1–12, 2017.

N. Gambino and R. Garner
The identity type weak factorization system.
Theoret. Comput. Sci., 409 (2008), pp. 94–109

D. R. Licata and R. Harper.
2-Dimensional Directed Type Theory.
Electronic Notes in Theoretical Computer Science, 276 (2011), pp. 263-289.

A. Nyuts.
Towards a Directed Homotopy Type Theory based on 4 Kinds of Variance.
MA thesis. KU Leuven, 2015.

P. R. North
Type theoretic weak factorization systems.
PhD Thesis. University of Cambridge. 2017.

P. R. North
Towards a directed homotopy type theory.
arXiv:1807.10566. 2018

E. Riehl and M. Shulman.
A type theory for synthetic ∞-categories.
Higher Structures, 1(1):116-193, 2017.

M. Warren.
Directed Type Theory
Talk at IAS, 10 April 2013.