A higher structure identity principle

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Outline

Motivation

Equivalence principle

Two equivalent structures must share the same structural properties.

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Generalizing *Univalent categories and the Rezk completion*, Ahrens, Kapulkin, Shulman, 2015.

Outline

Lower structure identity principles in UF

Theorem

Given two mere propositions P and Q,

$$(P =_{\mathsf{hProp}} Q) = (P \leftrightarrows Q)$$

Corollary

If P and Q are equivalent mere propositions, then they share the same structural properties.

For any $X : \mathsf{hProp} \vdash S(X) : \mathcal{U}$,

$$(P \leftrightarrows Q) \rightarrow (S(P) = S(Q)).$$

Lower structure identity principles in UF

Theorem (Coquand-Danielsson 2013)

Given two monoids M and N,

$$(M =_{\mathsf{Mon}} N) = (M \cong N).$$

Corollary

If M and N are isomorphic monoids, then they share the same structural properties.

For any $X : \mathsf{Mon} \vdash S(X) : \mathcal{U}$,

$$(M \cong N) \to (S(M) = S(N)).$$

Outline

A structure identity principle for categories in UF

A **category** $\mathscr{C}$ is given by

- a type $\mathscr{C}_0 : \mathscr{U}$ of **objects**
- for any $a, b : \mathscr{C}_0$, a set $\mathscr{C}(a, b) : \mathscr{U}$ of **morphisms**
- operations: identity & composition

$$1_a : \mathscr{C}(a, a)$$
$$(\circ)_{a,b,c} : \mathscr{C}(b, c) \to \mathscr{C}(a, b) \to \mathscr{C}(a, c)$$

- axioms: unitality & associativity

$$1 \circ f = f \qquad f \circ 1 = f \qquad (h \circ g) \circ f = h \circ (g \circ f)$$

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$$1 \circ f = f \qquad f \circ 1 = f \qquad (h \circ g) \circ f = h \circ (g \circ f)$$

A **univalent category** is a category $\mathscr{C}$ such that

$$(a = b) \to (a \cong b)$$

is an equivalence for all $a, b : \mathscr{C}_0$.

A structure identity principle for categories in UF

Theorem (Ahrens-Kapulkin-Shulman 2015)

For categories $\mathscr{C}$ and $\mathscr{D}$, let $\mathscr{C} \simeq \mathscr{D}$ denote the type of functors from $\mathscr{C}$ to $\mathscr{D}$ that are equivalences.

If $\mathscr{C}$ and $\mathscr{D}$ are univalent, then

$$(\mathscr{C} =_{\mathsf{UCat}} \mathscr{D}) = (\mathscr{C} \simeq \mathscr{D}).$$

Corollary

If $\mathscr{C}$ and $\mathscr{D}$ are equivalent univalent categories, then they share the same structural properties.

For any $X : \mathsf{UCat} \vdash P(X) : \mathscr{U}$,

$$(\mathscr{C} \simeq \mathscr{D}) \to (P(\mathscr{C}) = P(\mathscr{D})).$$

Goal

Conjecture

Given a signature $\mathscr{L}$, and two $\mathscr{L}$-univalent $\mathscr{L}$-structures M and N, then

$$(M = N) = (M \simeq_{\mathscr{L}} N)$$

Need notions of

- signatures $\mathscr{L}$
- $\mathscr{L}$-structures
- $\mathscr{L}$-equivalence of $\mathscr{L}$-structures
- $\mathscr{L}$-univalence of $\mathscr{L}$-structures

Outline

Two-level type theory

Working in the two-level type theory of Annenkov-Capriotti-Kraus.

- Universes $\mathscr{U} \hookrightarrow \mathscr{U}^s$
- $\mathscr{U}$ implements univalent type theory.
- Every type $T : \mathscr{U}^s$ is equipped with a strict equality type $a \equiv_T b$ with the usual rules for the identity type, but which also satisfies UIP.

First-order logic with dependent sorts

Inverse category

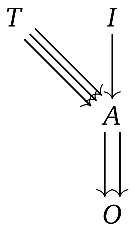
An *inverse category* is a strict category $\mathscr{I}$ and a function $\rho : \mathscr{I} \to \mathsf{Nat}^{\mathsf{op}}$ whose fibers are discrete.

The *height* of an inverse category $(\mathscr{I}, \rho)$ is the maximum value of ρ.

Signatures

Signatures are inverse categories of finite height.

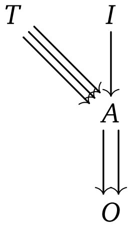
$\mathscr{L}_{\mathrm{cat}} :=$



$\mathcal{L}_{\text{cat}}$-structures

We can define the data of a category $\mathscr{C}$ to be

- A type $\mathscr{C}O : \mathscr{U}$
- A family $\mathscr{C}A : \mathscr{C}O \times \mathscr{C}O \to \mathscr{U}$
- A family $\mathscr{C}I : \prod_{(x:\mathscr{C}O)} \mathscr{C}A(x,x) \to \mathscr{U}$
- A family $\mathscr{C}T : \prod_{(x,y,z:\mathscr{C}O)} \mathscr{C}A(x,y) \to \mathscr{C}A(y,z) \to \mathscr{C}A(x,z) \to \mathscr{U}$



Here:

- Think of $\mathscr{C}I$, $\mathscr{C}T$ as the *predicates* ‘is an identity’, ‘is a composite’.
- $\mathcal{L}_{\text{cat}}$-*univalence* will imply that $\mathscr{C}I$, $\mathscr{C}T$ are pointwise propositions.
- $\mathcal{L}_{\text{cat}}$-*univalence* will imply that $\mathscr{C}A$ is pointwise a set.
- $\mathcal{L}_{\text{cat}}$-*univalence* will imply that $\mathscr{C}O$ is a 1-type.

Equality

To the data, we add axioms such as

- "There is a composite of every composable pair of arrows."

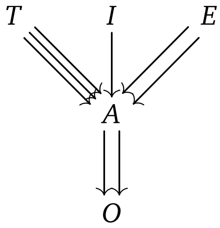
$$\forall(x,y,z:O).\forall(f:A(x,y)).\forall(g:A(y,z)).\exists(h:A(x,z)).T(x,y,z,f,g,h)$$

- "Composites are unique."

$$\forall(x,y,z:O).\forall(f:A(x,y)).\forall(g:A(y,z)).\forall(h,h':A(x,z)).\\ T(x,y,z,f,g,h)\to T(x,y,z,f,g,h')\to(h=h')$$

So we need to add an equality 'predicate':

$\mathscr{L}_{\mathrm{cat+E}} :=$



$\mathcal{L}_{\mathrm{cat+E}}$-structures

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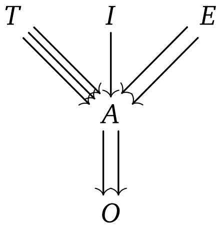
- A family $\mathscr{C}I : \prod_{(x:\mathscr{C}O)} \mathscr{C}A(x,x) \to \mathscr{U}$
- A family $\mathscr{C}T : \prod_{(x,y,z:\mathscr{C}O)} \mathscr{C}A(x,y) \to \mathscr{C}A(y,z) \to \mathscr{C}A(x,z) \to \mathscr{U}$
- A family $\mathscr{C}E : \prod_{(x,y:\mathscr{C}O)} \mathscr{C}A(x,y) \to \mathscr{C}A(x,y) \to \mathscr{U}$



Here:

- $\mathcal{L}_{\mathrm{cat+E}}$-*univalence* will imply that $\mathscr{C}E$ is a proposition.
- $\mathcal{L}_{\mathrm{cat+E}}$-*univalence* + axioms making E into an equivalence relation and congruence will imply that $(f = g) = \mathscr{C}E(f,g)$.

1-univalent FOLDS-categories

A 1-univalent FOLDS-category consists of an $\mathscr{L}_{\mathrm{cat+E}}$-structure

- $\mathscr{C}O : \mathscr{U}$
- $\mathscr{C}A : \mathscr{C}O \times \mathscr{C}O \to \mathscr{U}$
- $\mathscr{C}I : \prod_{(x:\mathscr{C}O)} \mathscr{C}A(x,x) \to \mathscr{U}$
- $\mathscr{C}T : \prod_{(x,y,z:\mathscr{C}O)} \mathscr{C}A(x,y) \to \mathscr{C}A(y,z) \to \mathscr{C}A(x,z) \to \mathscr{U}$
- $\mathscr{C}E : \prod_{(x,y:\mathscr{C}O)} \mathscr{C}A(x,y) \to \mathscr{C}A(x,y) \to \mathscr{U}$

such that

- $\mathscr{C}I_x(f)$, $\mathscr{C}T_{x,y,z}(f,g,h)$, and $\mathscr{C}E_{x,y}(f,g)$ are propositions
- $\mathscr{C}A(x,y)$ is a set,
- $\mathscr{C}E_{x,y}(f,g) = (f = g)$,

and the axioms of a category are satisfied.

Lemma

The type of 1-univalent FOLDS-cats is equivalent to the type of (pre)categories.

Univalent FOLDS-categories

Goal

To state the univalence condition

$$(a = b) = (a \cong b)$$

for categories in terms of the the FOLDS structure.

Given $a, b : \mathscr{C}O$, we can define an isomorphism $a \cong b$ using the Yoneda Lemma:

- For each $x : \mathscr{C}O$, an equality $\phi_{x\bullet} : \mathscr{C}A(x, a) = \mathscr{C}A(x, b)$.
- For each $x, y : \mathscr{C}O$, $f : \mathscr{C}A(x, y)$, $g : \mathscr{C}A(y, a)$, and $h : \mathscr{C}A(x, a)$, we have

$$\mathscr{C}T_{x,y,a}(f, g, h) = \mathscr{C}T_{x,y,b}(f, \phi_{y\bullet}(g), \phi_{x\bullet}(h))$$

$(\phi_{y\bullet}(g) \circ f = \phi_{x\bullet}(g \circ f))$

This is a bit ad hoc and not symmetric.

FOLDS isomorphism for categories

Instead, can define $a \cong b$ to consist of the following equalities between all the types of our signature with a and b substituted in *all* possible ways:

- For each $x : \mathscr{C}O$, an equality $\phi_{x\bullet} : \mathscr{C}A(x,a) = \mathscr{C}A(x,b)$.
- For each $z : \mathscr{C}O$, an equality $\phi_{\bullet z} : \mathscr{C}A(a,z) = \mathscr{C}A(b,z)$.
- An equality $\phi_{\bullet\bullet} : \mathscr{C}A(a,a) = \mathscr{C}A(b,b)$.
- The following equalities for all appropriate w, x, y, z, f, g, h:

$$T_{x,y,a}(f,g,h) = T_{x,y,b}(f, \phi_{y\bullet}(g), \phi_{x\bullet}(h))$$
$$T_{x,a,z}(f,g,h) = T_{x,b,z}(\phi_{x\bullet}(f), \phi_{\bullet z}(g), h)$$
$$T_{a,z,w}(f,g,h) = T_{b,z,w}(\phi_{\bullet z}(f), g, \phi_{\bullet w}(h))$$
$$T_{x,a,a}(f,g,h) = T_{x,b,b}(\phi_{x\bullet}(f), \phi_{\bullet\bullet}(g), \phi_{x\bullet}(h))$$
$$T_{a,x,a}(f,g,h) = T_{b,x,b}(\phi_{\bullet x}(f), \phi_{x\bullet}(g), \phi_{\bullet\bullet}(h))$$
$$T_{a,a,x}(f,g,h) = T_{b,b,x}(\phi_{\bullet\bullet}(f), \phi_{\bullet x}(g), \phi_{\bullet x}(h))$$
$$T_{a,a,a}(f,g,h) = T_{b,b,b}(\phi_{\bullet\bullet}(f), \phi_{\bullet\bullet}(g), \phi_{\bullet\bullet}(h))$$

$$I_{a,a}(f) = I_{b,b}(\phi_{\bullet\bullet}(f))$$
$$E_{x,a}(f,g) = E_{x,b}(\phi_{x\bullet}(f), \phi_{x\bullet}(g))$$
$$E_{a,x}(f,g) = E_{b,x}(\phi_{\bullet x}(f), \phi_{\bullet x}(g))$$
$$E_{a,a}(f,g) = E_{b,b}(\phi_{\bullet\bullet}(f), \phi_{\bullet\bullet}(g))$$

"Everything above a, b thinks that a and b are the same."

Univalent FOLDS categories

Theorem

In any 1-univalent FOLDS category, the type of isomorphisms $a \cong b$ just defined is equivalent to the type of ordinary isomorphisms $a \cong b$.

Definition

A univalent FOLDS category is a 1-univalent FOLDS category such that for all $a, b : \mathscr{C}O$, the canonical map

$$(a = b) \to (a \cong b)$$

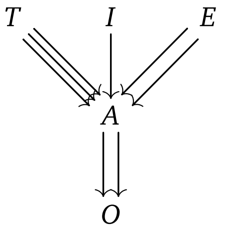
is an equivalence.

Theorem

A 1-univalent FOLDS category is univalent if and only if its corresponding precategory is a univalent category.

Univalence

- 1-univalence can be defined in the same way.
- For example, we required $\mathscr{C}T_{x,y,a}(f,g,h)$ to be a proposition.
- For any $c,d : \mathscr{C}T_{x,y,a}(f,g,h)$, everything above c,d 'thinks' c and d are the same, trivially.
- So $c \cong d$, and $\mathscr{C}T$ being univalent means that $(c = d) = (c \cong d)$.
- $\mathscr{C}T$ being univalent means that each $\mathscr{C}T_{x,y,a}(f,g,h)$ is a proposition.
- ...



Categorical equivalences

For univalent FOLDS categories $\mathscr{C}, \mathscr{D}$, we had an equivalence.

$$(\mathscr{C} = \mathscr{D}) \simeq (\mathscr{C} \simeq \mathscr{D})$$

We can also generalize categorical equivalences:

- A *very surjective morphism* $F : \mathscr{C} \twoheadrightarrow \mathscr{D}$ of $\mathscr{L}_{\mathrm{cat+E}}$-structures consists of surjections
  - $F_O : \mathscr{C}O \twoheadrightarrow \mathscr{D}O$
  - $F_A : \prod_{x,y:\mathscr{C}O} \mathscr{C}A(x,y) \twoheadrightarrow \mathscr{D}A(F_O x, F_O y)$
  - $F_T : \prod_{x,y,z:\mathscr{C}O, f:\mathscr{C}A(x,y), g:\mathscr{C}A(y,z), h:\mathscr{C}A(x,z)} \mathscr{C}T(f,g,h) \twoheadrightarrow \mathscr{D}T(F_A f, F_A g, F_A h)$
  - ...

Theorem

If $\mathscr{C}$ and $\mathscr{D}$ are univalent,

$$(\mathscr{C} \twoheadrightarrow \mathscr{D}) \simeq (\mathscr{C} \simeq \mathscr{D})$$

Outline

The framework

We can generalize this to any inverse category.

We generalize it further.

Signatures

We define signatures inductively to be a sequence of strict categories $\mathsf{Sig} : \mathsf{Nat}^s \to \mathsf{sCat}$.

- $\mathsf{Sig}(0) := \mathscr{U}$
- $\mathsf{Sig}(n+1)$ consists of a signature Z of level 0, and for every Z-structure $S : Z \to \mathscr{U}$, a *derivative* $DS : \mathsf{Sig}(n)$.

- $\mathscr{L}_{\text{cat+E}} : \mathsf{Sig}(2)$
- The 0-part is $*$.
- The derivative gives us a 1-signature for every type O. The 0-part of this 1-signature is $O \times O$.

We can also define structures, isomorphism, univalence, and very surjective morphisms following the example of categories.

Envisioned results

Almost-theorem

Consider $\mathscr{L}$-structures M, N for some signature $\mathscr{L}$ such that M is univalent. Then

$$(M \twoheadrightarrow N) = (M = N)$$

Conjecture

For a signature $L : \mathsf{Sig}(n)$, the type of univalent L-structures is of h-level $n+1$.

Thank you!