Coinductive control of inductive data types

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based on:
Coinductive control of inductive data types, North & Péroux
Measuring data types, Mulder, North & Péroux
and work in progress

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Outline

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Overview

Theorem (Mulder-N.-Péroux)

The category of algebras of an *accessible, lax symmetric monoidal* endofunctor on a *locally presentable, symmetric monoidal closed* category is enriched over the category of coalgebras of the same endofunctor. For any such category $\mathcal{C}$, we get a functor

$$\mathsf{Endo}_{\mathsf{alsm}}(\mathcal{C}) \to \mathsf{EnrCat}.$$

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Examples

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Gain

More "initial algebras" (e.g. generalized W-types)

Previous work on coalgebraic enrichment

Univeral measuring coalgebra (Sweedler, Wraith 1968)

For k-algebras A and B, there is a k-coalgebra $\underline{\mathrm{Alg}}(A, B)$

- which underlies an enrichment of k-algebras in k-coalgebras
- whose *set-like elements* are in bijection with $\mathrm{Alg}(A, B)$.

Analogues

- Anel-Joyal 2013 (dg-algebras)
- Hyland-Lopez Franco-Vasilakopoulou 2017 (monoids)
- Vasilakopoulou 2019 ($\mathcal{V}$-categories)
- Péroux 2022 (∞-algebras of an ∞-operad)
- McDermott-Rivas-Uustalu 2022 (monads)
- North-Péroux 2023 (algebras of endofunctors)
- ...

Motivation: inductive types

- In functional programming, most types are defined *inductively*.
- Categorically: initial alg of polynomial endofunctor (W-type)

Example: $\mathbb{N}$

- $\mathbb{N}$ is the initial algebra of the endofunctor $X \mapsto X + 1$ (on Set)
- The terminal coalgebra is $\mathbb{N}^\infty$
- This functor satisfies the hypotheses of our theorem.

Example: lists in a set A

- The set of lists in A is the initial algebra of $X \mapsto 1 + A \times X$.
- The terminal coalgebra is the set of *streams* in A.
- With a commutative monoid structure on A, this functor satisfies the hypotheses of our theorem.

Outline

Measuring in general

Fix a *locally presentable, symmetric monoidal closed* category $\mathcal{C}$ and an *accessible, lax symmetric monoidal* endofunctor F.

Definition: measure

For algebras $(A, \alpha), (B, \beta)$ a *measure* $(A, \alpha) \to (B, \beta)$ is a coalgebra (C, χ) together with a morphism $\phi : C \to \underline{\mathcal{C}}(A, B)$ satisfying:

$$\begin{array}{ccccc} & & FC \xrightarrow{F(\phi)} F(\underline{\mathcal{C}}(A, B)) & \longrightarrow & \underline{\mathcal{C}}(FA, FB) \\ C & \overset{\chi}{\nearrow} & & & \downarrow \beta \\ & \underset{\phi}{\searrow} & \underline{\mathcal{C}}(A, B) & \xrightarrow{\quad\alpha\quad} & \underline{\mathcal{C}}(FA, B) \end{array}$$

The *universal measure* $\underline{\mathrm{Alg}}(A, B)$ is the terminal one.

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Theorem (N.-Péroux)

The universal measure $\underline{\mathrm{Alg}}(A, B)$ always exists, and these are the hom-coalgebras of an enrichment of Alg_F in CoAlg_F.

Measuring for the natural numbers

Measuring

For algebras A, B, a *measure* $A \to B$ is a coalgebra C together with a function $C \to A \to B$ such that

- $f_c(0_A) = 0_B$ for all $c \in C$;
- $f_c(a+1) = 0_B$ for all $[\![c]\!] = 0$ and for all $a \in A$;
- $f_c(a+1) = f_{c-1}(a) + 1$ for $[\![c]\!] \geqslant 1$ and for all $a \in A$.

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What is this?

Set-like elements in general

Definition: set-like elements

The *set-like elements* are

$$\square \to \underline{\mathsf{Alg}}(A, B) \qquad \text{in } \mathsf{CoAlg}(F)$$

i.e., elements of $\mathsf{Alg}(A, B)$.

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Example

$$\mathsf{Alg}(\mathbb{N}, A) \cong *$$
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So denote the elements of $\underline{\mathsf{Alg}}(\mathbb{N}, A)$ by

- f_0
- f_1

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So denote the elements of $\underline{\mathrm{Alg}}(\mathbb{N}, A)$ by

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Definition

So we call elements of the underlying of $\underline{\mathsf{Alg}}(A, B)$ *n-partial algebra homomorphisms*.

What are the non-set-like elements?

- Let $\mathbb{n}$ denote the quotient of $\mathbb{N}$ by $m = n$ for all $m \geqslant n$.
- Let $\mathbb{n}^\circ$ denote the subobject of $\mathbb{N}^\infty$ consisting of $\{0, \ldots, n\}$.

Example

$$\mathrm{Alg}(\mathbb{n}, A) \cong \begin{cases} * & \text{if } n_A = m_A \text{ for all } m \geqslant n; \\ \varnothing & \text{otherwise.} \end{cases}$$

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- So there is at least always an n-partial homomorphism out of n (which is unique).

What can we do with this? Generalize W-types, i.e., initial algebras.

Definition: C-initial objects

For a coalgebra C, a *C-initial algebra* is an algebra A such that for all other algebras B there is a unique

$$C \to \underline{\mathsf{Alg}}(A, B).$$

Examples

For the natural-numbers endofunctor:

- $\mathbb{N}$ is the $\mathbb{N}^\infty$*-initial algebra* (i.e., initial wrt total algebra homs)
- $\mathbb{n}$ is the $\mathbb{n}^\circ$*-initial algebra* (i.e., initial wrt partial algebra homs)

Examples

On a locally presentable symmetric monoidal category $\mathcal{C}$:

- (id) The identity endofunctor
- (A) The constant endofunctor at fixed commutative monoid A
- (GF) The composition of two instances
- ($F \otimes G$) The tensor of two instances ($\mathcal{C}$ closed)
- ($F + G$) The coproduct of an instance F and an 'F-module' G
- (id^A) The exponential id^A at object A ($\mathcal{C}$ cartesian closed)
- (W-type) The polynomial endofunctor associated to a morphism $f : X \to Y$, given a commutative monoid structure on Y and an oplax symmetric monoidal structure on the preimage functor $f^{-1} : C \to \mathrm{Set}$ ($\mathcal{C} := \mathrm{Set}$)
- (d.e.s.) A discrete equational system of Leinster (monoidal structure on $\mathcal{C}$ is cocartesian, $\mathcal{C}$ has binary products that preserve filtered colimits)

Outline

Overview and background

Endofunctors

Work in progress: generalization

Proof sketch of main theorem[1]

Convolution algebra

We get a functor

$$[-,-] : \mathsf{CoAlg}^{\mathsf{op}} \times \mathsf{Alg} \to \mathsf{Alg}$$
$$(C,\chi),\ (A,\alpha) \mapsto (\underline{\mathcal{C}}(C,A), ?)$$

where ? is the composite

$$F\underline{\mathcal{C}}(C,A) \to \underline{\mathcal{C}}(FC,FA) \xrightarrow{\alpha^*\chi_*} \underline{\mathcal{C}}(C,A).$$

Then we use the adjoint functor theorem to get an enriched hom

$$\underline{\mathsf{Alg}}(-,-) : \mathsf{Alg}^{\mathsf{op}} \times \mathsf{Alg} \to \mathsf{CoAlg}.$$

[1] $\mathcal{C}$ a locally presentable, symmetric monoidal closed category; F an accessible, lax symmetric monoidal endofunctor

Generalizations/analogues: more convolution algebras[3]

Let F be lax symmetric monoidal, G colax symmetric monoidal and colax closed.

- For $F, G : \mathcal{C} \to \mathcal{D}$: (F, G)-dialgebras[2] are enriched in (G, F)-dialgebras.

From

$$F\underline{\mathcal{C}}(C, A) \to \underline{\mathcal{C}}(FC, FA) \xrightarrow{\alpha^* \chi_*} \underline{\mathcal{C}}(GC, GA) \to G\underline{\mathcal{C}}(C, A).$$

[2] objects are pairs $(X \in \mathcal{C}, \delta : FX \to GX)$

[3] all categories locally presentable, symmetric monoidal closed; all functors accessible

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- For $F : \mathcal{C} \to \mathcal{E}$, $G : \mathcal{D} \to \mathcal{E}$: $F \downarrow G$ is enriched in $G \downarrow F$.

From

$$F\underline{\mathcal{C}}(C, A) \to \underline{\mathcal{C}}(FC, FA) \xrightarrow{\alpha^* \chi_*} \underline{\mathcal{C}}(GC', GA') \to G\underline{\mathcal{C}}(C', A').$$

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- For $F : \mathcal{C} \to \mathcal{E}$, $G : \mathcal{D} \to \mathcal{E}$: $F \downarrow G$ is enriched in $G \downarrow F$.
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Summary

We have

- that algebras are enriched in coalgebras (under certain hypotheses)
- an interpretation as notion of partial algebra homomorphism (especially in the case $\mathbb{N}$)
- many examples
- a more refined notion of initial algebra
- a generalization ...

Thank you!