The Univalence Principle

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Outline

1. What is / why (homotopy) type theory?
2. What is / why univalence?
3. Univalent categories
4. Univalent categories II

Why type theory?

- Homotopy type theory is the logic of homotopy theory
- Equality in the type theory corresponds to homotopy
  - We don't have recourse to 'classical' equality
  - We are forced to do everything up to homotopy (unless we can figure out a way to do it fibrewise)
- Proofs are computer-checkable.

What is type theory?

- Type theory is a language for mathematics, akin to category theory.
- Sentences are of the following form:
  - $a_1 : A_1, \ldots, a_n : A_n \vdash B(a_1, \ldots, a_n)$ type
  - $a_1 : A_1, \ldots, a_n : A_n \vdash b(a_1, \ldots, a_n) : B(a_1, \ldots, a_n)$
- We conflate mathematical objects and mathematical statements.
  - $n : \mathbb{N} \vdash \mathsf{isEven}(n)$ type
  - $n : \mathbb{N} \vdash e(n) : \mathsf{isEven}(2n)$
  - $X : U \vdash \mathsf{isContr}(X)$ type
  - $X : U \vdash c(X) : \mathsf{isContr}(CX)$ type

Interpretations of type theory

- Examples:
  - $n : \mathbb{N} \vdash \mathsf{isEven}(n)$ type
  - $n : \mathbb{N} \vdash e(n) : \mathsf{isEven}(2n)$
  - $X : U \vdash \mathsf{isContr}(X)$ type
  - $X : U \vdash c(X) : \mathsf{isContr}(CX)$ type
  - $n : \mathbb{N} \vdash \mathsf{Vect}_{\mathsf{n}}(\mathbb{N})$ type
  - $n : \mathbb{N} \vdash \mathsf{o}(n) : \mathsf{Vect}_{\mathsf{n}}(\mathbb{N})$ type
- There are many interpretations of dependent type theory:

	Contexts	**Types**	**Terms**
Logical	hypotheses	predicates	proofs
Set theoretic	indices	indexed sets	sections
Homotopical	base space	total space	sections

Type formers

- We can define the natural numbers, booleans, the circle, (dependent) functions, (dependent) products and coproducts as initial objects in the following way.

Natural numbers

$$\frac{}{\vdash \mathbb{N}\ \text{type}} \qquad \frac{}{\vdash \mathrm{o} : \mathbb{N}} \qquad \frac{\vdash x : \mathbb{N}}{\vdash sx : \mathbb{N}}$$

$$\frac{x : \mathbb{N} \vdash D(x)\ \text{type} \qquad \vdash z : D(\mathrm{o}) \qquad x : \mathbb{N}, y : D(x) \vdash \sigma(y) : D(sx)}{x : \mathbb{N} \vdash d(x) : D(x)}$$

$$\vdash d(\mathrm{o}) \equiv z : D(\mathrm{o}) \qquad x : \mathbb{N} \vdash \sigma(d(x)) \equiv d(sx) : D(sx)$$

The identity type

Identity type

$$\frac{\vdash A \text{ type} \qquad \vdash a, b : A}{\vdash a =_A b} \qquad \frac{\vdash A \text{ type} \qquad \vdash a : A}{\vdash r_a : a =_A a}$$

$$\frac{\vdash A \text{ type} \qquad x, y : A,\ p : x =_A y \vdash D(p) \text{ type} \qquad x : A \vdash \rho(x) : D(r_x)}{x, y : A,\ p : x =_A y \vdash d(p) : D(p)}$$

$$x : A \vdash \rho(x) \equiv d(r_x) : D(r_x)$$

- For a homotopy theorist, there are two important things in homotopy theory: $=$ (homotopy) and $\vdash$ (fibration).

Outline

The univalence axiom

- We can prove for any $x : B \vdash E(x)$ type and any $b =_B c$, that $E(b) =_U E(c)$.
- (A type $x : B \vdash E(x)$ type is equivalently a function $E : B \to U$.)
- What is an equality $E(b) =_U E(c)$?
- We can separately define a type of equivalences $E \simeq F$ to consist of terms (f, g, h, i) where $f : E \leftrightarrows F : g$, $h(x) : fgx =_F x$, $i(y) : y =_E gfy$ for all $x : F, y : E$.[1]

The univalence axiom (Voevodsky)

For any two types E, F, the canonical function

$$E =_U F \to E \simeq F$$

is an equivalence.

[1] This is a lie: we actually want an adjoint equivalence.

Univalence in general

Synthetic vs. analytic equalities

In type theory, we always have a (*synthetic*) equality type between $a, b : T$

$$a =_T b.$$

Depending on the type T, we might have a type of "*analytic* equalities"

$$a \cong b.$$

A "univalence principle" for this T and this $\cong$ states that

$$(a =_T b) \to (a \cong_T b)$$

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The univalence *axiom* is a univalence principle where $T = U$ and $\cong_T$ is set to $\simeq$, equivalence between types.

Identicals and indiscernibilites

Identity of indiscernibles

Leibniz: two things are equal when they are *indiscernible* (have the same properties).

$$(a = b) \leftarrow \left(\forall P. P(a) \leftrightarrow P(b)\right)$$

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$$(a =_T b) \leftrightarrow \left(\prod_{P:T \to \mathcal{U}} P(a) \simeq P(b) \right)$$

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- This holds in MLTT.

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$$(a =_T b) \leftrightarrow \left(\prod_{P:T\to\mathcal{U}} P(a) \simeq P(b) \right)$$

- This holds in MLTT.
- Given a 'univalence principle' $(a =_T b) \simeq (a \cong b)$, we would find a *structure identity principle* (in the sense of Aczel):

$$(a \cong b) \to \left(\prod_{P:T\to\mathcal{U}} P(a) \simeq P(b) \right).$$

Goal

Our goal

To define a large class of (higher) *structures* and a notion of *equivalence* between them validating a univalence principle. This then automatically validates a structure identity principle.

Using ideas from:

- *First Order Logic with Dependent Sorts*, Makkai, 1995.
- *Univalent categories and the Rezk completion*, Ahrens, Kapulkin, Shulman, 2015.

Outline

h-levels

We can stratify (some) types into h-levels.

0: T is *contractible* if

$$\mathsf{isContr}(T) := \Sigma_{c:T}\Pi_{y:T}\, c =_T y$$

1: T is a *proposition* if

$$\mathsf{isProp}(T) := \Pi_{x,y:T}\mathsf{isContr}(x =_T y)$$

2: T is a *set* if

$$\mathsf{isSet}(T) := \Pi_{x,y:T}\mathsf{isProp}(x =_T y)$$

3: T is a *groupoid* if

$$\mathsf{isGpd}(T) := \Pi_{x,y:T}\mathsf{isSet}(x =_T y)$$

$n+1$: T is of *h-level* $n+1$ if

$$\mathsf{ishlevel}(n+1)(T) := \Pi_{x,y:T}\mathsf{ishlevel}(n)(x =_T y)$$

Categories

Definition (Ahrens, Kapulkin, Shulman 2015)

A *category* $\mathscr{C}$ consists of

- $\mathsf{ob}\mathscr{C} : U$
- $x, y : \mathsf{ob}\mathscr{C} \vdash \hom(x, y) : \mathsf{Set}$
- $x : \mathsf{ob}\mathscr{C} \vdash 1_x : \hom(x, x)$
- $x, y, z : \mathsf{ob}\mathscr{C}, f : \hom(x, y), g : \hom(y, z) \vdash g \circ f : \hom(x, z)$
- $x, y : \mathsf{ob}\mathscr{C}, f : \hom(x, y) \vdash \mathsf{rUni}(f) : 1_y \circ f =_{\hom(x,y)} f$
- $x, y : \mathsf{ob}\mathscr{C}, f : \hom(x, y) \vdash \mathsf{lUni}(f) : f \circ 1_x =_{\hom(x,y)} f$
- $w, x, y, z : \mathsf{ob}\mathscr{C}, f : \hom(w, x), g : \hom(x, y), h : \hom(y, z) \vdash$ $\mathsf{ass}(f, g, h) : (h \circ g) \circ f =_{\hom(w,z)} h \circ (g \circ f)$

Univalent categories

Definition (Ahrens, Kapulkin, Shulman 2015)

A category $\mathscr{C}$ is a *univalent* category if for all $x, y : \mathsf{ob}\mathscr{C}$, the canonical function

$$(x =_{\mathsf{ob}\mathscr{C}} y) \to (x \cong y)$$

is an equivalence.

Theorem (Ahrens, Kapulkin, Shulman 2015)

Given two *univalent* categories $\mathscr{C}$ and $\mathscr{D}$, the canonical function

$$(C =_{\mathsf{UCat}} D) \to (C \simeq D)$$

is an equivalence.

A language for invariant properties

Michael Makkai, *Towards a Categorical Foundation of Mathematics*:

> *"The basic character of the Principle of Isomorphism is that of a* ***constraint on the language*** *of Abstract Mathematics; a welcome one, since it provides for the separation of sense from non-sense."*

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Example

There are statements that can be made in set theory that distinguish the following two categories, but there are none in type theory (when interpreting them as univalent categories):

More precise goal

Our more precise goal

To define a large class of *univalent* (higher) *structures* and a notion of *equivalence* between them validating a univalence principle. This then automatically validates a structure identity principle.

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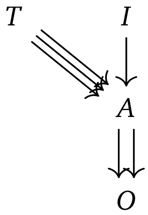
$\mathscr{L}_{\text{cat}}$-structures

Instead of thinking of $1_\bullet, \circ$ as functions, we can think of them as relations I, T. We can define a category $\mathscr{C}$ to be:

- $O : U$
- $x, y : O \vdash A(x, y) : U$
- $x : O, f : A(x, x) \vdash I_x(f) : U$
- $x, y, z : O, f : A(x, y), g : A(y, z), h : A(x, z) \vdash T_{x,y,z}(f, g, h) : U$



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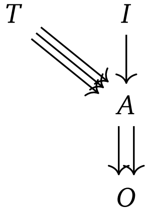

We want to add axioms such as

$$\forall(x,y,z : O).\forall(f : A(x,y)).\forall(g : A(y,z)).\forall(h,h' : A(x,z)).$$
$$T_{x,y,z}(f,g,h) \to T_{x,y,z}(f,g,h') \to (h = h')$$

(composites are unique), so we add an equality 'predicate'.

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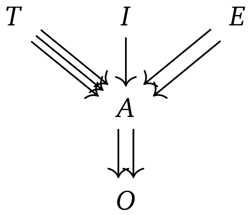
- $x, y : O, f, g : A(x,y) \vdash E_{x,y}(f,g) : U$



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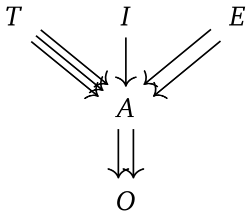

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(composites are unique), so we add an equality 'predicate'.

Signatures

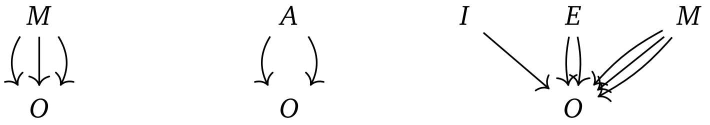
Inverse category

An *inverse category* is a strict category $\mathscr{I}$ and a function $\rho : \mathscr{I} \to \mathsf{Nat}^{\mathsf{op}}$ whose fibers are discrete.

The *height* of an inverse category $(\mathscr{I}, \rho)$ is the maximum value of ρ.

Signatures

Signatures are inverse categories of finite height.



$\mathscr{L}_{\mathsf{Magma}}$ $\mathscr{L}_{\mathsf{Proset}}$ $\mathscr{L}_{\mathsf{Group}}$

Structures

An $\mathscr{L}$-structure for a signature $\mathscr{L}$ is a ‘Reedy fibrant diagram’ $\mathscr{L} \to U$.

Indiscernibility

Definition

Given an $\mathscr{L}$-structure $M : \mathscr{L} \to U$, and an object S of $\mathscr{L}$, we say that two elements $x, y : MS$ are *indiscernible* if substituting x for y in any type that depends on (i.e. object with a morphism to) S produces equivalent types.

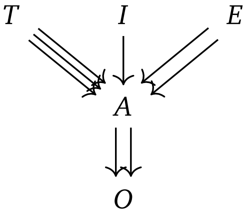
Definition

An $\mathscr{L}$-structure $M : \mathscr{L} \to U$ is *univalent* if for any $x, y : MS$, the type of indiscernibilities between x and y is equivalent to the type of equalities between x and y.

Example

Let $\mathscr{L}_{\text{cat}}$ be the signature for categories, and $\mathscr{C}$ a univalent $\mathscr{L}_{\text{cat}}$ structure.

- Any two terms $x : O, f : A(x,x) \vdash i, j : I_x(f)$ are indiscernible because there are no objects with a morphism to I. So each $I_x(f)$ is a proposition.
- Similarly, any two terms in $T_{x,y,z}(f,g,h)$ or $E_{x,y}(f,g)$ are indiscernible. So each $T_{x,y,z}(f,g,h)$, $E_{x,y}(f,g)$ is a proposition.



  - Two 'morphisms' $x, y : \text{ob}\mathscr{C} \vdash f, g : A(x,y)$ are indiscernible if (among other things) $E_{x,y}(\alpha, f) \cong E_{x,y}(\alpha, g)$ for all $\alpha : A(x,y)$. The axioms for E say (1) this implies $E_{x,y}(f,g)$, and (2) that $E_{x,y}(f,g)$ implies f and g are indiscernible (E is a congruence for E, T, I). Thus, $f = g$ is equivalent to $E_{x,y}(f,g)$.

Univalence at O

- The indiscernibilities between $a, b : \mathscr{C}O$ consist of
  1. $\phi_{x\bullet} : \mathscr{C}A(x,a) \simeq \mathscr{C}A(x,b)$ for each $x : \mathscr{C}O$
  2. $\phi_{\bullet z} : \mathscr{C}A(a,z) \simeq \mathscr{C}A(b,z)$ for each $z : \mathscr{C}O$
  3. $\phi_{\bullet\bullet} : \mathscr{C}A(a,a) \simeq \mathscr{C}A(b,b)$
  4. The following for all appropriate w, x, y, z, f, g, h:

$$T_{x,y,a}(f,g,h) \leftrightarrow T_{x,y,b}(f, \phi_{y\bullet}(g), \phi_{x\bullet}(h))$$
$$T_{x,a,z}(f,g,h) \leftrightarrow T_{x,b,z}(\phi_{x\bullet}(f), \phi_{\bullet z}(g), h)$$
$$T_{a,z,w}(f,g,h) \leftrightarrow T_{b,z,w}(\phi_{\bullet z}(f), g, \phi_{\bullet w}(h))$$
$$T_{x,a,a}(f,g,h) \leftrightarrow T_{x,b,b}(\phi_{x\bullet}(f), \phi_{\bullet\bullet}(g), \phi_{x\bullet}(h))$$
$$T_{a,x,a}(f,g,h) \leftrightarrow T_{b,x,b}(\phi_{\bullet x}(f), \phi_{x\bullet}(g), \phi_{\bullet\bullet}(h))$$
$$T_{a,a,x}(f,g,h) \leftrightarrow T_{b,b,x}(\phi_{\bullet\bullet}(f), \phi_{\bullet x}(g), \phi_{\bullet x}(h))$$
$$T_{a,a,a}(f,g,h) \leftrightarrow T_{b,b,b}(\phi_{\bullet\bullet}(f), \phi_{\bullet\bullet}(g), \phi_{\bullet\bullet}(h))$$

$$I_a(f) \leftrightarrow I_b(\phi_{\bullet\bullet}(f))$$
$$E_{x,a}(f,g) \leftrightarrow E_{x,b}(\phi_{x\bullet}(f), \phi_{x\bullet}(g))$$
$$E_{a,x}(f,g) \leftrightarrow E_{b,x}(\phi_{\bullet x}(f), \phi_{\bullet x}(g))$$
$$E_{a,a}(f,g) \leftrightarrow E_{b,b}(\phi_{\bullet\bullet}(f), \phi_{\bullet\bullet}(g))$$

Univalent $\mathscr{L}_{\text{cat}}$-structures continued

Proposition

The type of indiscernibilities between $a, b : O$ is equivalent to $a \cong b$.

Proof.

The isomorphisms $\phi_{x\bullet} : A(x, a) \cong A(x, b)$ are natural by

$$T_{x,y,a}(f, g, h) \longleftrightarrow T_{x,y,b}(f, \phi_{y\bullet}(g), \phi_{x\bullet}(h))$$

(saying $\phi_{y\bullet}(g) \circ f = \phi_{x\bullet}(g \circ f)$). The rest of the data is redundent.

Thus, in a univalent $\mathscr{L}_{\text{cat}}$-structure, $(a = b) \simeq (a \cong b)$.

Theorem

Univalent $\mathscr{L}_{cat}$-structures are equivalent to the univalent categories of Ahrens-Kapulkin-Shulman.

Categorical equivalences

A categorial equivalence arises as a very surjective morphism.

A *very surjective morphism* or *equivalence* $F : \mathscr{C} \simeq \mathscr{D}$ of $\mathscr{L}_{\mathrm{cat+E}}$-structures consists of surjections

- $FO : \mathscr{C}O \twoheadrightarrow \mathscr{D}O$
- $FA : \mathscr{C}A(x,y) \twoheadrightarrow \mathscr{D}A(Fx,Fy)$ for every $x,y : \mathscr{C}O$
- $FT : \mathscr{C}T(f,g,h) \twoheadrightarrow \mathscr{D}T(Ff,Fg,Fh)$ for all $f : \mathscr{C}A(x,y), g : \mathscr{C}A(y,z), h : \mathscr{C}A(x,z)$
- $FE : \mathscr{C}E(f,g) \twoheadrightarrow \mathscr{D}E(Ff,Fg)$ for all $f,g : \mathscr{C}A(x,y)$
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Equivalences in general

Definition (equivalence)

An *equivalence* $M \simeq N$ between two $\mathscr{L}$-structures is a very split-surjective morphism $M \to N$.

Theorem

Given two univalent $\mathscr{L}$-structures M and N,

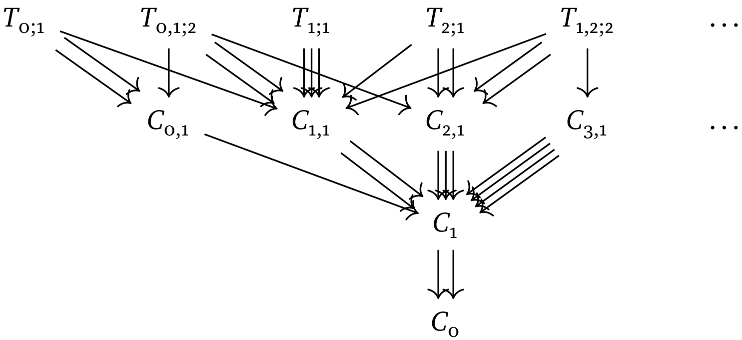
$$(M = N) \simeq (M \simeq N).$$

Theorem

For a signature $L : \mathsf{Sig}(n)$, the type of univalent L-structures is of h-level $n+1$.

Opetopic bicategories

The signature is the following plus a unary predicate $U_{n,1}$ and a binary predicate $E_{n,1}$ on each $C_{n,1}$.



Univalence makes each $T_{...}$ a proposition, each $C_{n,1}$ a set with equality given by $E_{n,1}$, C_1 the objects of a univalent category, and equality in C_0 equivalent to adjoint equivalence.

Summary

For every signature $\mathscr{L}$, we have

- a notion of indiscernibility within each sort,
- a notion of univalent structures,
- a notion of equivalence,
- a univalence theorem,
- and thus a (higher) structure identity principle.

Summary

For every signature $\mathcal{L}$, we have

- a notion of indiscernibility within each sort,
- a notion of univalent structures,
- a notion of equivalence,
- a univalence theorem,
- and thus a (higher) structure identity principle.

The paper includes examples of

- $\dagger$-categories,
- presheaves,
- profunctors,
- semi-displayed categories,
- bicategories,
- ...

Further work

- Drop the splitness condition for certain structures
- Extend to infinite structures
- Formulate an analogue to the Rezk completion
- Translate the theory into one about structures which can include functions
- Explore examples

Thank you!