

Proving the equivalence principle in Univalent Foundations

Paige Randall North

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Outline

- 1 Motivation
- 2 Lower equivalence principles in univalent foundations
- 3 First-order logic with dependent sorts (FOLDS) for lower structures
- 4 FOLDS categories
- 5 Double categories

Last time

We have:

Equality principle

$$\prod_{x,y:A} (x = y) \rightarrow \prod_{f:A \rightarrow B} (f(x) = f(y))$$

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e.g.

$$\prod_{G,H:Grp} (G \cong H) \rightarrow \prod_{B:Grp \rightarrow Latt} B(G) \cong B(H)$$

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What's missing is a tighter relationship between equality and notions of equivalence.

Different notions of equality

Synthetic vs. analytic equalities

In MLTT, we always have a (*synthetic*) equality type between $a, b : T$

$$a =_T b.$$

Depending on the type T , we might have a type of “*analytic* equalities”

$$a \cong b.$$

A “univalence principle” for this T and this $\cong$ states that

$$(a =_T b) \rightarrow (a \cong b)$$

is an equivalence.

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The univalence *axiom* in type theory states that

$$S =_{\mathcal{U}} T \rightarrow S \simeq T$$

is an equivalence.

Identicals and indiscernibilites

Identity of indiscernibles

Leibniz: two things are equal when they are *indiscernible* (have the same properties).

$$(a = b) \leftarrow (\forall P. P(a) \leftrightarrow P(b))$$

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- This holds in MLTT.
- Given a ‘univalence principle’ $(a =_T b) \simeq (a \cong b)$, we would find a *structure identity principle* (in the sense of Aczel):

$$(a \cong b) \rightarrow \left(\prod_{P:T \rightarrow \mathcal{U}} P(a) \simeq P(b) \right).$$

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h-levels

We can stratify (some) types into h-levels.

0: T is *contractible* if

$$\text{isContr}(T) := \Sigma_{c:T} \Pi_{y:T} c =_T y$$

1: T is a *proposition* if

$$\text{isProp}(T) := \Pi_{x,y:T} \text{isContr}(x =_T y)$$

2: T is a *set* if

$$\text{isSet}(T) := \Pi_{x,y:T} \text{isProp}(x =_T y)$$

3: T is a *groupoid* if

$$\text{isGpd}(T) := \Pi_{x,y:T} \text{isSet}(x =_T y)$$

$n + 1$: T is of *h-level* $n + 1$ if

$$\text{ishlevel}(n + 1)(T) := \Pi_{x,y:T} \text{ishlevel}(n)(x =_T y)$$

Propositions

Assuming the Univalence Axiom:

$$(S =_{\mathcal{U}} T) \simeq (S \simeq T)$$

for every type S, T :

Theorem (univalence for propositions)

Given two propositions P and Q ,

$$(P =_{\text{Prop}} Q) \simeq (P \leftrightarrow Q).$$

Interlude: subsets

Given a function $p : T \rightarrow \text{Prop}$, we can form the type $\Sigma_{t:T} p\ t$.

Theorem

There is a natural function

$$\iota : \Sigma_{t:T} p\ t \rightarrow T$$

and for $s, t : \Sigma_{t:T} p\ t$ we have

$$s =_{\Sigma_{t:T} p\ t} t \quad \simeq \quad \iota s =_T \iota t$$

Thus, we call such Σ -types *subsets* and ι the *inclusion*.

Example: isProp is a function $\mathcal{U} \rightarrow \text{Prop}$ so Prop is a subset of $\mathcal{U}$. Similarly, Set , Gpd , etc. are subsets of $\mathcal{U}$.

Sets

Theorem (univalence for sets)

Given two sets S and T ,

$$(S =_{\text{Set}} T) \simeq (S \cong T).$$

Theorem ('structure identity principle' for structures sets), Coquand-Danielsson

Given terms S, T of a type $\mathcal{S}$ of sets with structure (groups, monoids, etc),

$$(S =_{\mathcal{S}} T) \simeq (S \cong T)$$

Categories

Theorem (univalence for categories), Ahrens-Kapulkin-Shulman

Given two *univalent* categories C and D ,

$$(C =_{\mathbf{UCat}} D) \simeq (C \simeq D).$$

Definition

A category C is *univalent* if $(x =_{\mathbf{Ob}(C)} y) \simeq (x \cong y)$.

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Example



Indiscernibility

- One way to think about isomorphism in a category between two objects is as *indiscernibility*.
- In a univalent category, two objects are equal iff they are indiscernible.
- In order to generalize univalence for *univalent* categories, we generalized *indiscernibility*.

Magmas

Magmas

A *magma* is a set M and a binary operation $M \times M \rightarrow M$.

There are two notions of ‘sameness’ for elements m, n of a magma:

1. Equality: $m =_M n$

2. Indiscernibility:

$$\prod_{x,y:M} (mx = nx) \times (xm = xn) \times ((xy = m) \leftrightarrow (xy = n))$$

This produces two notions of equivalence of magmas:

1. $M \cong_e N$ if there are morphisms $f : M \hookrightarrow N : g$ respecting the operation such that gfm is *equal* to m for all $m : M$ and likewise for fgn
2. $M \cong_i N$ if there are morphisms $f : M \hookrightarrow N : g$ respecting the operation such that gfm is *indiscernible* from m for all $m : M$ and likewise for fgn

Preorders and topological spaces

Preorders

A *preorder* is a set P and a reflexive, transitive relation $\leq: P \times P \rightarrow \text{Prop}$. Two elements p, q of a preorder P are *indiscernible* if

$$\prod_{x:P} (p \leq x \leftrightarrow q \leq x) \times (x \leq p \leftrightarrow x \leq q) \times (p \leq p \leftrightarrow q \leq q)$$

or, equivalently, if $p \leq q \times q \leq p$.

We get two notions of equivalence of preorders:

1. $P \cong_e Q$
2. $P \cong_i Q$

A lower structure identity principle in UF

Theorem (univalence for magmas with $\cong_e$)

Given two magmas M, N ,

$$(M =_{\text{Mag}} N) \simeq (M \cong_e N).$$

- This is a special case of univalence for sets with structure (Coquand-Danielsson)
- The same holds for preorders with $\cong_e$.

Another lower structure identity principle in UF?

Univalence with $\cong_i$

Q: Can we hope for the same with $\cong_i$?

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A: No: for example, the projection $U : \mathbf{Mag} \rightarrow \mathbf{Set}$ would then take an equivalence $M \cong_i N$ to an equality/equivalence $M \cong_e N$.

For example, let **1** be the poset whose underlying set has one element, and let **2** be the poset whose underlying set has two elements a and b for which $a \leq b$ and $b \leq a$.



Another lower structure identity principle in UF?

Univalence with $\cong_i$

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A: Yes: if we identify equality and indiscernibility.

For example, let **1** be the poset whose underlying set has one element, and let **2** be the poset whose underlying set has two elements a and b for which $a \leq b$ and $b \leq a$.



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Goal

Our goal

To define a large class of (higher) *univalent structures* and a notion of *equivalence* between them validating a univalence principle. This then automatically validates a structure identity principle.

Using indiscernibility for the notions of

- univalent
- equivalence

Joint work with Ahrens, Shulman, Tsementzis. arXiv:[2004.06572](https://arxiv.org/abs/2004.06572)

First-order logic with dependent sorts (Makkai)

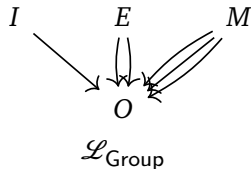
Inverse category

An *inverse category* is a strict category $\mathcal{I}$ and a function $\rho : \mathcal{I} \rightarrow \mathbf{Nat}^{\text{op}}$ whose fibers are discrete.

The *height* of an inverse category $(\mathcal{I}, \rho)$ is the maximum value of ρ .

Signatures

Signatures are inverse categories of finite height.



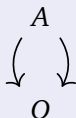
Structures

An $\mathcal{L}$ -structure is a Reedy-fibrant functor from $\mathcal{L}$ into $\mathcal{U}$.

$\mathcal{L}_{\text{Proset}}$ -structures

An $\mathcal{L}_{\text{Proset}}$ -structure S is

1. A type SO ,
2. A type $SA(x,y)$ for every $x,y : O$ (meaning $x \leq y$)



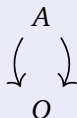
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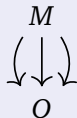
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$\mathcal{L}_{\text{Magma}}$ -structures

An $\mathcal{L}_{\text{Magma}}$ -structure S is

1. A type SO ,
2. A type $SM(x,y,z)$ for every $x,y,z : O$ (meaning z is the product of x and y)



We can impose axioms on these structures.

Indiscernibilities

Indiscernibilities between O -elements of $\mathcal{L}_{\text{Proset}}$ -structures

An indiscernibility between two terms $p, q : SO$ consists of

- $\prod_{x:SO} SA(p, x) \cong SA(q, x)$
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Indiscernibilities between O -elements of $\mathcal{L}_{\text{Magma}}$ -structures

An indiscernibility between two terms $m, n : SO$ consists of

- $\prod_{x,y:SO} SM(m, x, y) \cong SM(n, x, y)$
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Indiscernibilities at the top-level

Indiscernibilities between A-elements of $\mathcal{L}_{\text{Proset}}$ -structures

An indiscernibility between two terms $a, b : SA(p, q)$ consists of

- -

so all terms of $a, b : SA(p, q)$ are (trivially) indiscernible.

Definition (univalent structure)

A structure M of a signature $\mathcal{L}$ is *univalent* if the type of indiscernibilities between any two terms of any one sort is equivalent to the type of equalities between them.

In particular, this means that all of the top-level sorts are propositions, and all of the next-level sorts are sets.

Univalent structures

Proposition

A $\mathcal{L}_{\text{Proset}}$ -structure S is univalent when each $p \leq q$ is a proposition and $(p = q) \rightarrow (p \leq q) \times (q \leq p)$ is an equivalence - in other words, when A is a poset.

Proposition

A $\mathcal{L}_{\text{Magma}}$ -structure S is univalent when each $SM(m, n, p)$ is a proposition and $(m = n) \rightarrow \prod_{x, y: M} (mx = nx) \times (xm = xn) \times ((xy = m) \leftrightarrow (xy = n))$ is an equivalence.

Proposition

A topological space T is univalent when $(x = y) \rightarrow \prod_{U \text{ open in } T} (x \in U \leftrightarrow y \in U)$ is an equivalence - in other words, T is a T_0 space.

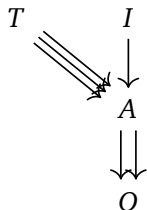
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$\mathcal{L}_{\text{cat}}$ -structures

We can define the data of a category $\mathcal{C}$ to be

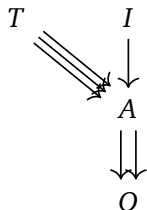
- A type $\mathcal{C}O : \mathcal{U}$
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- A family $\mathcal{C}T : \prod_{(x,y,z:\mathcal{C}O)} \mathcal{C}A(x,y) \rightarrow \mathcal{C}A(y,z) \rightarrow \mathcal{C}A(x,z) \rightarrow \mathcal{U}$



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We want to add axioms such as

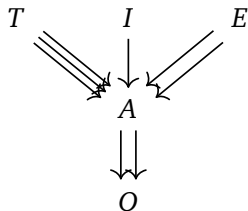
$$\forall (x,y,z : O). \forall (f : A(x,y)). \forall (g : A(y,z)). \forall (h, h' : A(x,z)). \\ T(x,y,z,f,g,h) \rightarrow T(x,y,z,f,g,h') \rightarrow (h = h')$$

(composites are unique), so we add an equality ‘predicate’.

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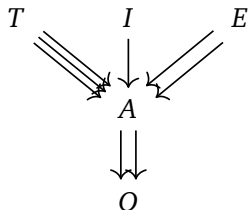
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- A family $\mathcal{C}E : \prod_{(x,y:\mathcal{C}O)} \mathcal{C}A(x,y) \rightarrow \mathcal{C}A(x,y) \rightarrow \mathcal{U}$



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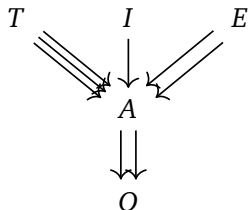
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- A family $\mathcal{C}E : \prod_{(x,y:\mathcal{C}O)} \mathcal{C}A(x,y) \rightarrow \mathcal{C}A(x,y) \rightarrow \mathcal{U}$



We want to add axioms such as

$$\forall (x,y,z : O). \forall (f : A(x,y)). \forall (g : A(y,z)). \forall (h, h' : A(x,z)). \\ T(x,y,z,f,g,h) \rightarrow T(x,y,z,f,g,h') \rightarrow E(h,h')$$

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Univalent $\mathcal{L}_{\text{cat}}$ -structures

- Every two elements of $\mathcal{C}I_x(f)$, $\mathcal{C}E_{x,y}(f,g)$, or $\mathcal{C}T_{x,y,z}(f,g,h)$ are indiscernible
 - so each of these types should be a proposition.
- The axioms making E a congruence for T and I make $\mathcal{C}E(f,g)$ the type of indiscernibilities between $f,g : \mathcal{C}A(x,y)$
 - so we should have $(f = g) = \mathcal{C}E(f,g)$, making each $\mathcal{C}A(x,y)$ a set.
- The indiscernibilities between $a,b : \mathcal{C}O$ consist of
 1. $\phi_{x\bullet} : \mathcal{C}A(x,a) \simeq \mathcal{C}A(x,b)$ for each $x : \mathcal{C}O$
 2. $\phi_{\bullet z} : \mathcal{C}A(a,z) \simeq \mathcal{C}A(b,z)$ for each $z : \mathcal{C}O$
 3. $\phi_{\bullet\bullet} : \mathcal{C}A(a,a) \simeq \mathcal{C}A(b,b)$
 4. The following for all appropriate w,x,y,z,f,g,h :

$$T_{x,y,a}(f,g,h) \leftrightarrow T_{x,y,b}(f, \phi_{y\bullet}(g), \phi_{x\bullet}(h))$$

$$I_{a,a}(f) \leftrightarrow I_{b,b}(\phi_{\bullet\bullet}(f))$$

$$T_{x,a,z}(f,g,h) \leftrightarrow T_{x,b,z}(\phi_{x\bullet}(f), \phi_{\bullet z}(g), h)$$

$$E_{x,a}(f,g) \leftrightarrow E_{x,b}(\phi_{x\bullet}(f), \phi_{x\bullet}(g))$$

$$T_{a,z,w}(f,g,h) \leftrightarrow T_{b,z,w}(\phi_{\bullet z}(f), g, \phi_{\bullet w}(h))$$

$$E_{a,x}(f,g) \leftrightarrow E_{b,x}(\phi_{\bullet x}(f), \phi_{\bullet x}(g))$$

$$T_{x,a,a}(f,g,h) \leftrightarrow T_{x,b,b}(\phi_{x\bullet}(f), \phi_{\bullet\bullet}(g), \phi_{x\bullet}(h))$$

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Univalent $\mathcal{L}_{\text{cat}}$ -structures continued

Proposition

The type of indiscernibilities between $a, b : \mathcal{C}O$ is equivalent to $a \cong b$.

Proof.

The isomorphisms $\phi_{x\bullet} : \mathcal{C}A(x, a) \cong \mathcal{C}A(x, b)$ are natural by

$$\mathcal{C}T_{x,y,a}(f, g, h) \leftrightarrow \mathcal{C}T_{x,y,b}(f, \phi_{y\bullet}(g), \phi_{x\bullet}(h))$$

(saying $\phi_{y\bullet}(g) \circ f = \phi_{x\bullet}(g \circ f)$). The rest of the data is redundant.

Thus, in a univalent $\mathcal{L}_{\text{cat}}$ -structure, $(a = b) \simeq (a \cong b)$.

Theorem

Univalent $\mathcal{L}_{\text{cat}}$ -structures are equivalent to the univalent categories of Ahrens-Kapulkin-Shulman.

Categorical equivalences

Theorem (univalence for univalent categories) (Ahrens-Kapulkin-Shulman)

Given univalent categories $\mathcal{C}, \mathcal{D}$,

$$(\mathcal{C} = \mathcal{D}) \simeq (\mathcal{C} \simeq \mathcal{D})$$

A categorial equivalence arises as a very surjective morphism.

A *very surjective morphism* or *equivalence* $F : \mathcal{C} \simeq \mathcal{D}$ of $\mathcal{L}_{\text{cat}+\mathbf{E}}$ -structures consists of surjections

- $FO : \mathcal{C}O \rightarrow \mathcal{D}O$
- $FA : \mathcal{C}A(x,y) \rightarrow \mathcal{D}A(Fx,Fy)$ for every $x,y : \mathcal{C}O$
- $FT : \mathcal{C}T(f,g,h) \rightarrow \mathcal{D}T(Ff,Fg,Fh)$ for all $f : \mathcal{C}A(x,y), g : \mathcal{C}A(y,z), h : \mathcal{C}A(x,z)$
- $FE : \mathcal{C}E(f,g) \rightarrow \mathcal{D}E(Ff,Fg)$ for all $f,g : \mathcal{C}A(x,y)$
- $FI : \mathcal{C}I(f) \rightarrow \mathcal{D}I(Ff)$ for all $f : \mathcal{C}A(x,x)$

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Given univalent categories $\mathcal{C}, \mathcal{D}$,

$$(\mathcal{C} = \mathcal{D}) \simeq (\mathcal{C} \simeq \mathcal{D})$$

A categorial equivalence arises as a very surjective morphism.

A *very surjective morphism* or *equivalence* $F : \mathcal{C} \simeq \mathcal{D}$ of **univalent** $\mathcal{L}_{\text{cat}+\mathbf{E}}$ -structures consists of surjections

- $FO : \mathcal{C}O \rightarrow \mathcal{D}O$
- $FA : \mathcal{C}A(x,y) \rightarrow \mathcal{D}A(Fx,Fy)$ for every $x,y : \mathcal{C}O$
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Equivalences in general

Definition (equivalence)

An *equivalence* $M \simeq N$ between two $\mathcal{L}$ -structures is a very surjective morphism $M \rightarrow N$.

Theorem

Given two univalent $\mathcal{L}$ -structures M and N ,

$$(M = N) \simeq (M \simeq N).$$

Theorem

For a signature $L : \text{Sig}(n)$, the type of univalent L -structures is of h -level $n + 1$.

Example: magmas

Equivalences of univalent magmas

An equivalence of magmas N, P consists of surjections

- $FO : NO \rightarrow PO$
- $FM : NM(x, y, z) \rightarrow PM(Fx, Fy, Fz)$

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Summary

For every signature $\mathcal{L}$, we have

- a notion of indiscernibility within each sort,
- a notion of univalent structures,
- a notion of equivalence,
- a univalence theorem,
- and thus a (higher) structure identity principle.

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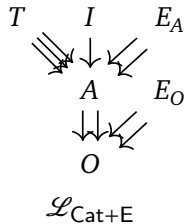
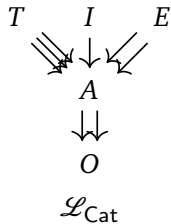
The paper includes examples of

- $\dagger$ -categories,
- presheaves,
- profunctors,
- semi-displayed categories,
- bicategories,
- ...

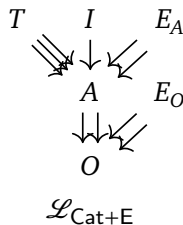
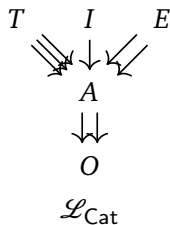
Outline

- 1 Motivation
- 2 Lower equivalence principles in univalent foundations
- 3 First-order logic with dependent sorts (FOLDS) for lower structures
- 4 FOLDS categories
- 5 Double categories**

Categories and set-categories

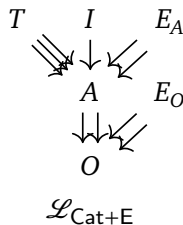
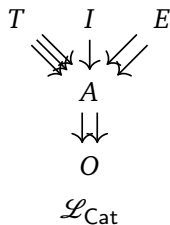


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- Different notions of equivalence are appropriate at different times.

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- Bicategorical equivalence

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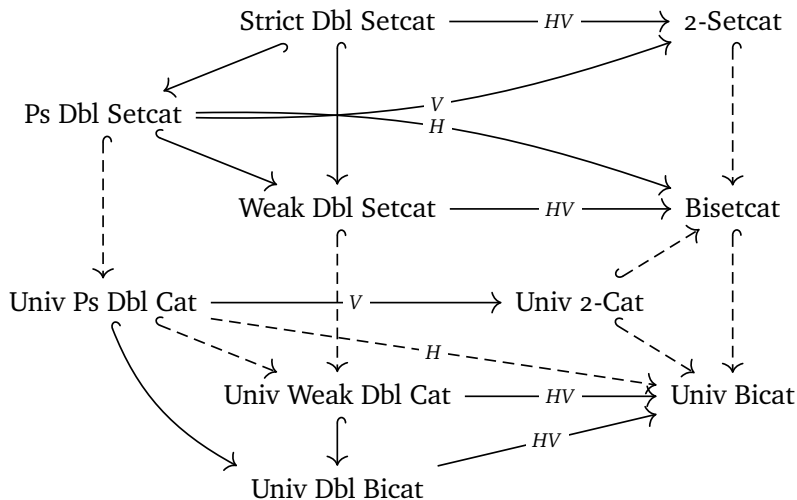
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We can give different definitions of bicategory for each.

Double categories, formalized in UniMath¹



¹jww Rasekh, van der Weide, Ahrens, arXiv:2402.05265

Thank you!