Coinductive control of inductive data types

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based on:
Coinductive control of inductive data types, North & Péroux
Measuring data types, Mulder, North & Péroux
and work in progress

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Outline

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Overview and background

Endofunctors

Work in progress: generalization

Overview

Theorem (Mulder-N.-Péroux)

The category of algebras of an *accessible, lax symmetric monoidal* endofunctor on a *locally presentable, symmetric monoidal closed* category is enriched over the category of coalgebras of the same endofunctor. For any such category $\mathcal{C}$, we get a functor

$$\mathsf{Endo}_{\mathsf{alsm}}(\mathcal{C}) \to \mathsf{EnrCat}.$$

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Examples

There are many examples, including polynomial endofunctors with extra structure.

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Gain

More "initial algebras" (e.g. generalized W-types)

Previous work on coalgebraic enrichment

Univeral measuring coalgebra (Sweedler, Wraith 1968)

For k-algebras A and B, there is a k-coalgebra $\underline{\mathrm{Alg}}(A, B)$

- which underlies an enrichment of k-algebras in k-coalgebras
- whose *set-like elements*[1] are in bijection with $\mathrm{Alg}(A, B)$.

Taking $B := k$, one gets the *dual* $\underline{\mathrm{Alg}}(A, k)$ of A.

[1] those $c \in \underline{\mathrm{Alg}}(A, B)$ s.t. $\Delta c = c \otimes c$ and $\epsilon(c) = 1_A$

Previous work on coalgebraic enrichment

Analogues

- Fox 1976 (monoids)
- Batchelor 2000 (modules)
- Anel-Joyal 2013 (dg-algebras)
- Hyland-Lopez Franco-Vasilakopoulou 2017 (monoids)
- Vasilakopoulou 2019 ($\mathcal{V}$-categories)
- Péroux 2022 (∞-algebras of an ∞-operad)
- McDermott-Rivas-Uustalu 2022 (monads)
- North-Péroux 2023 (algebras of endofunctors)
- Banerjee-Kour 2024 (stable anti-Yetter Drinfeld modules)
- Aravantinos-Sotiropoulos - Vasilakopoulou (modules in double categories)
- ...

Motivation: inductive types

- In functional programming, most types are defined *inductively*.
- Categorically: initial alg of polynomial endofunctor (W-type)

Example: $\mathbb{N}$

- $\mathbb{N}$ is the initial algebra of the endofunctor $X \mapsto X + 1$ (on Set)
- The terminal coalgebra is $\mathbb{N}^{\infty}$
- This functor satisfies the hypotheses of our theorem.

Example: lists in a set A

- The set of lists in A is the initial algebra of $X \mapsto 1 + A \times X$.
- The terminal coalgebra is the set of *streams* in A.
- With a commutative monoid structure on A, this functor satisfies the hypotheses of our theorem.

Outline

Overview and background

Endofunctors

Work in progress: generalization

Measuring in general

Fix a *locally presentable, symmetric monoidal closed* category $\mathcal{C}$ and an *accessible, lax symmetric monoidal* endofunctor F.

Definition: measure

For algebras $(A, \alpha), (B, \beta)$ a *measure* $(A, \alpha) \to (B, \beta)$ is a coalgebra (C, χ) together with a morphism $\phi : C \to \underline{\mathcal{C}}(A, B)$ satisfying:

$$
\begin{array}{ccccccc}
 & & FC & \xrightarrow{F(\phi)} & F(\underline{\mathcal{C}}(A,B)) & \longrightarrow & \underline{\mathcal{C}}(FA, FB) \\
 & \overset{\chi}{\nearrow} & & & & & \big\downarrow \beta \\
C & & & & & & \\
 & \underset{\phi}{\searrow} & & & & & \\
 & & \underline{\mathcal{C}}(A,B) & & \xrightarrow{\quad\quad\alpha\quad\quad} & & \underline{\mathcal{C}}(FA, B)
\end{array}
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The *universal measure* $\underline{\mathrm{Alg}}(A, B)$ is the terminal one.

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Theorem (N.-Péroux)

The universal measure $\underline{\mathrm{Alg}}(A, B)$ always exists, and these are the hom-coalgebras of an enrichment of Alg_F in CoAlg_F.

Measuring for the natural numbers

Measuring

For algebras A, B, a *measure* $A \to B$ is a coalgebra C together with a function $C \to A \to B$ such that

- $f_c(0_A) = 0_B$ for all $c \in C$;
- $f_c(a+1) = 0_B$ for all $[\![c]\!] = 0$ and for all $a \in A$;
- $f_c(a+1) = f_{c-1}(a) + 1$ for $[\![c]\!] \geqslant 1$ and for all $a \in A$.

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The *universal measure* $\underline{\mathrm{Alg}}(A, B)$ is the terminal measure $A \to B$.

What is this?

Set-like elements in general

Definition: set-like elements

The *set-like elements* are

$$\mathbb{1} \to \underline{\mathsf{Alg}}(A, B) \qquad \text{in } \mathsf{CoAlg}(F)$$

i.e., elements of $\mathsf{Alg}(A, B)$.

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- The *points* of $\mathrm{Alg}(A, B)$ are total algebra homomorphisms $A \to B$.

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That is

- The *points* of $\mathrm{Alg}(A, B)$ are total algebra homomorphisms $A \to B$.
- If we're considering $(\mathrm{Set}, \times, *)$, the underlying set of $\mathbb{I}$ is $*$, so these are 'special' elements of the underlying set of $\mathrm{Alg}(A, B)$.

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where $\Box$ has underlying set $\{*\}$ such that $* - 1 = *$

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so f_* is a total algebra homomorphism

Definition: measuring

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Set-like elements for the natural numbers

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where $\mathbb{1}$ has underlying set $\{*\}$ such that $* - 1 = *$
so $\mathbb{1} \to \underline{\mathsf{Alg}}(A, B)$ is an element $* \in \underline{\mathsf{Alg}}(A, B)$ s.t. $* - 1 = *$
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that is, an element of $\mathsf{Alg}(A, B)$.

Example

$$\mathsf{Alg}(\mathbb{N}, A) \cong *$$

$$\underline{\mathsf{Alg}}(\mathbb{N}, A) \cong \mathbb{N}^{\infty}$$

What are the non-set-like elements?

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So denote the elements of $\underline{\mathrm{Alg}}(\mathbb{N}, A)$ by

- f_0
- f_1

  ...
- f_∞

Definition: measuring

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- $f_0(0) = 0_B$
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So denote the elements of $\underline{\mathrm{Alg}}(\mathbb{N}, A)$ by

- $f_0(n) = 0_A$
- $f_1(0) = 0_A; f_1(n+1) = 1_A$

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- f_∞

Definition: measuring

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  ...
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Definition: measuring

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- $f_\infty(0) = 0_B$
- $f_\infty(a+1) = f_\infty(a) + 1$

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Definition

So we call elements of the underlying set of $\underline{\mathrm{Alg}}(A, B)$ *n-partial algebra homomorphisms*.

What are the non-set-like elements?

- Let $\mathbb{n}$ denote the quotient of $\mathbb{N}$ by $m = n$ for all $m \geqslant n$.
- Let $\mathbb{n}^\circ$ denote the subobject of $\mathbb{N}^\infty$ consisting of $\{0, \ldots, n\}$.

Example

$$\mathrm{Alg}(\mathbb{n}, A) \cong \begin{cases} * & \text{if } n_A = m_A \text{ for all } m \geqslant n; \\ \varnothing & \text{otherwise.} \end{cases}$$

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Example

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$$\underline{\mathrm{Alg}}(\mathbb{n}, A) \cong \begin{cases} \mathbb{N}^\infty & \text{if } n_A = m_A \text{ for all } m \geqslant n; \\ \mathbb{n}^\circ & \text{otherwise.} \end{cases}$$

- So there is at least always an n-partial homomorphism out of n (which is unique).

What can we do with this?

Generalize W-types, i.e., initial algebras.

Definition: C-initial objects

For a coalgebra C, a *C-initial algebra* is an algebra A such that for all other algebras B there is a unique

$$C \to \underline{\mathrm{Alg}}(A, B).$$

Initial object

An initial object in a category $\mathcal{C}$ is an object A such that for all other algebras B there is a unique

$$* \to \mathcal{C}(A, B).$$

C-initial objects for the natural numbers

Examples

For the natural-numbers endofunctor:

- $\mathbb{N}$ is the $\mathbb{I}$-*initial algebra*
- $\mathbb{N}$ is the $\mathbb{N}^{\infty}$-*initial algebra*

C-initial objects for the natural numbers

Examples

For the natural-numbers endofunctor:

- $\mathbb{N}$ is the $\mathbb{I}$-*initial algebra*
- $\mathbb{N}$ is the $\mathbb{N}^\infty$-*initial algebra*

- $\mathbb{I}$- (or $\mathbb{N}^\infty$-) initial means initial with respect to total algebra homomorphisms

Theorem

$\mathbb{n}$ is the $\mathbb{n}^\circ$-*initial algebra*

- $\mathbb{n}^\circ$-initial means initial with respect to partial algebra homomorphisms

Examples

On a locally presentable symmetric monoidal category $\mathcal{C}$:

- (id) The identity endofunctor
- (A) The constant endofunctor at fixed commutative monoid A
- (GF) The composition of two instances
- ($F \otimes G$) The tensor of two instances ($\mathcal{C}$ closed)
- ($F + G$) The coproduct of an instance F and an 'F-module' G
- (id^A) The exponential id^A at object A ($\mathcal{C}$ cartesian closed)
- (W-type) The polynomial endofunctor associated to a morphism $f : X \to Y$, given a commutative monoid structure on Y and an oplax symmetric monoidal structure on the preimage functor $f^{-1} : C \to \mathrm{Set}$ ($\mathcal{C} := \mathrm{Set}$)
- (d.e.s.) A discrete equational system of Leinster (monoidal structure on $\mathcal{C}$ is cocartesian, $\mathcal{C}$ has binary products that preserve filtered colimits)

Outline

Proof sketch of main theorem[2]

Convolution algebra

We get a functor

$$[-,-] : \mathsf{CoAlg}^{\mathsf{op}} \times \mathsf{Alg} \to \mathsf{Alg}$$
$$(C,\chi),\ (A,\alpha) \mapsto (\underline{\mathcal{C}}(C,A), ?)$$

where ? is the composite

$$F\underline{\mathcal{C}}(C,A) \to \underline{\mathcal{C}}(FC,FA) \xrightarrow{\alpha^*\chi_*} \underline{\mathcal{C}}(C,A).$$

Then we use the adjoint functor theorem to get an enriched hom

$$\underline{\mathsf{Alg}}(-,-) : \mathsf{Alg}^{\mathsf{op}} \times \mathsf{Alg} \to \mathsf{CoAlg}.$$

[2] $\mathcal{C}$ a locally presentable, symmetric monoidal closed category; F an accessible, lax symmetric monoidal endofunctor

Generalizations/analogues: more convolution algebras[4]

Let F be lax symmetric monoidal, G colax symmetric monoidal and colax closed.

- For $F, G : \mathcal{C} \to \mathcal{D}$: (F, G)-dialgebras[3] are enriched in (G, F)-dialgebras.

From

$$F\underline{\mathcal{C}}(C, A) \to \underline{\mathcal{C}}(FC, FA) \xrightarrow{\alpha^* \chi_*} \underline{\mathcal{C}}(GC, GA) \to G\underline{\mathcal{C}}(C, A).$$

[3] objects are pairs $(X \in \mathcal{C}, \delta : FX \to GX)$

[4] all categories locally presentable, symmetric monoidal closed; all functors accessible

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- For $F, G : \mathcal{C} \to \mathcal{D}$: (F, G)-dialgebras[3] are enriched in (G, F)-dialgebras.
- For $F : \mathcal{C} \to \mathcal{E}$, $G : \mathcal{D} \to \mathcal{E}$: $F \downarrow G$ is enriched in $G \downarrow F$.

From

$$F\underline{\mathcal{C}}(C, A) \to \underline{\mathcal{C}}(FC, FA) \xrightarrow{\alpha^* \chi_*} \underline{\mathcal{C}}(GC', GA') \to G\underline{\mathcal{C}}(C', A').$$

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- For $F : \mathcal{C} \to \mathcal{E}$, $G : \mathcal{D} \to \mathcal{E}$: $F \downarrow G$ is enriched in $G \downarrow F$.
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Summary

We have

- that algebras are enriched in coalgebras (under certain hypotheses)
- an interpretation as notion of partial algebra homomorphism (especially in the case $\mathbb{N}$)
- many examples
- a more refined notion of initial algebra
- a generalization ...

Thank you!