Perspectives on the univalence principle

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Outline

1. Background on univalent foundations
2. The univalence principle[1]
3. Double categories[2]

[1] jww Ahrens, Shulman, Tsementzis
[2] jww Rasekh, van der Weide, Ahrens

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Different notions of equality

Synthetic vs. analytic equalities

In type theory with the equality type, we always have a ("synthetic") equality type between $a, b : D$

$$a =_D b.$$

Depending on the type D, we might also have a type of "analytic" equalities

$$a \simeq_D b.$$

A *univalence principle* for this D and this $\simeq_D$ states that

$$(a =_D b) \to (a \simeq_D b)$$

is an equivalence.

The univalence axiom

Voevodsky postulated a univalence principle for types.

The univalence axiom

The canonical function $(A =_{\mathsf{Type}} B) \to (A \simeq B)$ is an equivalence of types, for any types A and B.

Identicals and indiscernibilites

Identity of indiscernibles

Leibniz: two things are equal when they are *indiscernible* (have the same properties).

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$$(a =_D b) \leftrightarrow \left(\prod_{P:D\to\mathsf{Type}} P(a) \simeq P(b) \right)$$

- This holds in type theory.
- Given a univalence principle $(a =_D b) \simeq (a \simeq_D b)$, we find an *equivalence principle*:

$$(a \simeq_D b) \to \left(\prod_{P:D\to\mathsf{Type}} P(a) \simeq P(b) \right).$$

Univalent mathematics

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- For types A, B which are structured sets (groups, rings, etc),
$$(A =_{\mathrm{Grp}} B) \overset{UA}{\simeq} (A \simeq B) \simeq (A \cong B)$$
so everything respects isomorphism of groups (or rings, etc).[5]

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  so everything respects isomorphism of groups (or rings, etc).[5]
- For *univalent* categories A, B,

$$(A =_{\text{UCat}} B) \overset{UA}{\simeq} (A \simeq B) \simeq (A \simeq B)$$

  so everything respects equivalence of univalent categories.[6]

[5] Coquand-Danielsson 2013
[6] Ahrens-Kapulkin-Shulman 2015

Univalent mathematics

- Voevodsky dreamt of ‘univalent mathematics’ in which

  $$(A =_{\mathrm{D}} B) \simeq (A \simeq_{\mathrm{D}} B)$$

  where D is any type of mathematical object (propositions, sets, groups, categories, ∞-categories, etc) and $\simeq_{\mathrm{D}}$ is the appropriate notion of ‘sameness’ for that type of objects.
- This would give us an appropriate language in which to study D.

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- Equivalence principles are often the things we sweep under the rug in informal mathematics.
- In UF: they're a theorem.
- Formalization is appropriate for reasoning about large, unwieldy (higher categorical) structures.
- In UF: we have support.

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- Partially formalized in Agda with the two-level flag[9].

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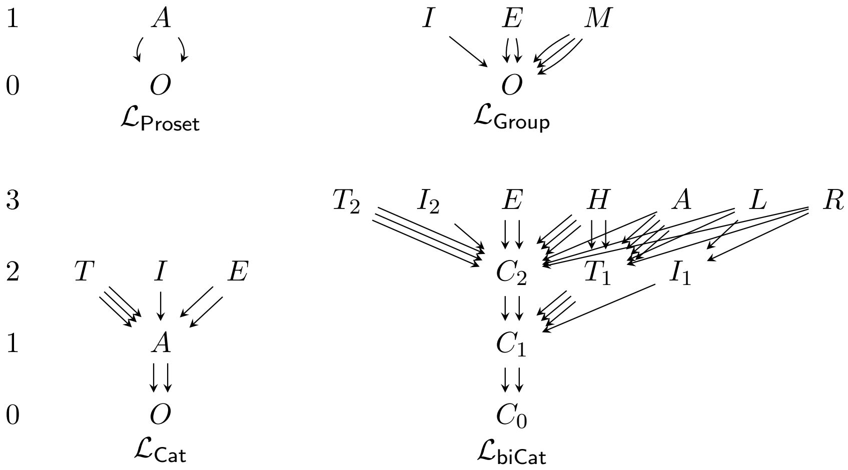
The univalence principle

- We realize Voevosky's dream for "finite" algebraic structures.
- Uses two-level type theory (homotopy type theory + formal meta level).
- Partially formalized in Agda with the two-level flag[9].
- Meta-theorem (and unwieldy), so provides recipe for formalization in UF-based systems.

[9] by Üsküplü

Signatures


1 A 0 O
$\mathcal{L}_{\mathsf{Proset}}$
I E M O
$\mathcal{L}_{\mathsf{Group}}$
3 2 T I E 1 A 0 O
$\mathcal{L}_{\mathsf{Cat}}$
T_2 I_2 E H A L R
C_2 T_1 I_1
C_1
C_0
$\mathcal{L}_{\mathsf{biCat}}$

Structures

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- In type theory, we define an $\mathcal{L}$**-structure** fiberwise.
- An $\mathcal{L}_{\mathsf{Cat}}$-structure $\mathcal{C}$ consists of:

$$\begin{tikzcd} T \arrow[dr, shift left] \arrow[dr] \arrow[dr, shift right] & I \arrow[d] & E \arrow[dl, shift left] \arrow[dl, shift right] \\ & A \arrow[d, shift left] \arrow[d, shift right] & \\ & O & \end{tikzcd}$$

$\mathcal{L}_{\mathsf{Cat}}$

- $\mathcal{C}O : \mathsf{Type}$
- $x, y : \mathcal{C}O \vdash \mathcal{C}A(x, y) : \mathsf{Type}$
- $x : \mathcal{C}O, f : \mathcal{C}A(x, x) \vdash \mathcal{C}I_x(f) : \mathsf{Type}$
- $x, y, z : \mathcal{C}O, f : \mathcal{C}A(x, y), g : \mathcal{C}A(y, z), h : \mathcal{C}A(x, z) \vdash \mathcal{C}T_{x,y,z}(f, g, h) : \mathsf{Type}$
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- An $\mathcal{L}_{\mathsf{Cat}}$-structure $\mathcal{C}$ consists of:

$$
\begin{tikzcd}
T \arrow[dr, shift left=2] \arrow[dr] \arrow[dr, shift right=2] & I \arrow[d] & E \arrow[dl, shift left=1] \arrow[dl, shift right=1] \\
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- Then we add axioms.

Level-wise equivalence

Proposition

For two $\mathcal{L}$-structures S, T,

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A levelwise equivalence $\mathcal{C} \cong_{\mathcal{L}_{\mathsf{Cat}}-\mathsf{Str}} \mathcal{D}$ consists of:

- $e_O : \mathcal{C}O \xrightarrow{\sim} \mathcal{D}O$
- $x, y : \mathcal{C}O \vdash e_A : \mathcal{C}A(x, y) \xrightarrow{\sim} \mathcal{D}(e_O x, e_O y)$
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But this is not an equivalence of categories.
And is it appropriate to call $\mathcal{C}, \mathcal{D}$ categories?

Indiscernibility

Definition

Given an $\mathcal{L}$-structure M, and an object S of $\mathcal{L}$, we say that two elements $x, y : MS$ are *indiscernible* if substituting x for y in any object of $\mathcal{L}$ that depends on (i.e. object with a morphism to) S produces equivalent types.

Definition

An $\mathcal{L}$-structure M is *univalent* if for any object S of $\mathcal{L}$, and any $x, y : MS$, the type of indiscernibilities between x and y is equivalent to the type of equalities between x and y.

Univalent $\mathcal{L}_{\mathrm{cat}}$ structures

Let $\mathcal{C}$ be a univalent $\mathcal{L}_{\mathrm{cat}}$ structure.

$$\begin{tikzcd}
T \arrow[dr, shift left=1.5] \arrow[dr] \arrow[dr, shift right=1.5] & I \arrow[d] & E \arrow[dl, shift left] \arrow[dl, shift right] \\
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- Any two terms $x : \mathcal{C}O, f : \mathcal{C}A(x,x) \vdash i, j : \mathcal{C}I_x(f)$ are indiscernible.

$\rightarrow$ Each $\mathcal{C}I_x(f)$ is a proposition.

$\rightarrow$ Similarly, each $\mathcal{C}T_{x,y,z}(f,g,h)$, $\mathcal{C}E_{x,y}(f,g)$ is a proposition.

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- In the axioms for a category, we have that E behaves like equality (is reflexive and a congruence for T, I, E.)

→ Univalence at A means that $f = g$ is equivalent to $\mathcal{C}E_{x,y}(f,g)$.

→ $\mathcal{C}A(x,y)$ is a set.

Univalent $\mathcal{L}_{\mathrm{cat}}$ structures

- The indiscernibilities between $a, b : \mathcal{C}O$ consist of
  - $\phi_{x\bullet} : \mathcal{C}A(x,a) \cong \mathcal{C}A(x,b)$ for each $x : \mathcal{C}O$
  - $\phi_{\bullet z} : \mathcal{C}A(a,z) \cong \mathcal{C}A(b,z)$ for each $z : \mathcal{C}O$
  - $\phi_{\bullet\bullet} : \mathcal{C}A(a,a) \cong \mathcal{C}A(b,b)$
  - The following for all appropriate w, x, y, z, f, g, h:

$$\mathcal{C}T_{x,y,a}(f,g,h) \leftrightarrow \mathcal{C}T_{x,y,b}(f, \phi_{y\bullet}(g), \phi_{x\bullet}(h))$$
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- But this an isomorphism in the usual categorical sense.

$\rightarrow$ Univalence at O means that $x = y$ is equivalent to $x \cong y$.

$\rightarrow$ cf. Complete Segal spaces

The right notion of equivalence

Main theorem

For two *univalent* $\mathcal{L}$-structures S, T,

$$(S =_{\mathcal{L}-\mathsf{Str}} T) \simeq (S \cong_{\mathcal{L}-\mathsf{Str}} T) \simeq (S \cong^*_{\mathcal{L}-\mathsf{Str}} T) \simeq (S \twoheadrightarrow T)$$

where $\cong^*_{\mathcal{L}-\mathsf{Str}}$ denotes levelwise equivalence up to indiscernbility and $\twoheadrightarrow$ denotes a very split surjective morphism.

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Very surjective morphisms of $\mathcal{L}_{\mathrm{cat}}$-structures

A *very surjective morphism* or *equivalence* $F : \mathcal{C} \simeq \mathcal{D}$ of $\mathcal{L}_{\mathrm{cat}}$-structures consists of surjections

- $FO : \mathcal{C}O \twoheadrightarrow \mathcal{D}O$
- $FA : \mathcal{C}A(x, y) \twoheadrightarrow \mathcal{D}A(Fx, Fy)$ for every $x, y : \mathcal{C}O$
- $FT : \mathcal{C}T(f, g, h) \twoheadrightarrow \mathcal{D}T(Ff, Fg, Fh)$ for all $f : \mathcal{C}A(x, y), g : \mathcal{C}A(y, z), h : \mathcal{C}A(x, z)$
- $FE : \mathcal{C}E(f, g) \twoheadrightarrow \mathcal{D}E(Ff, Fg)$ for all $f, g : \mathcal{C}A(x, y)$
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The right notion of equivalence

Main theorem

For two *univalent* $\mathcal{L}$-structures S, T,

$$(S =_{\mathcal{L}-\mathsf{Str}} T) \simeq (S \cong_{\mathcal{L}-\mathsf{Str}} T) \simeq (S \cong^*_{\mathcal{L}-\mathsf{Str}} T) \simeq (S \twoheadrightarrow T)$$

where $\cong^*_{\mathcal{L}-\mathsf{Str}}$ denotes levelwise equivalence up to indiscernbility and $\twoheadrightarrow$ denotes a very split surjective morphism.

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Summary

For every signature $\mathcal{L}$, we have

- a notion of structure,
- a notion of indiscernibility within each sort,
- a notion of univalent structures,
- a notion of equivalence,
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The paper includes examples of

- †-categories,
- profunctors,
- bicategories,
- opetopic bicategories,
- ...

Outline

1. Background on univalent foundations
2. The univalence principle[10]
3. Double categories[11]

[10] jww Ahrens, Shulman, Tsementzis
[11] jww Rasekh, van der Weide, Ahrens

Categories and set-categories

$$\begin{tikzcd}
T \arrow[dr, shift left=2] \arrow[dr] \arrow[dr, shift right=2] & I \arrow[d] & E \arrow[dl, shift left=1] \arrow[dl, shift right=1] \\
 & A \arrow[d, shift left=1] \arrow[d, shift right=1] & \\
 & O &
\end{tikzcd}$$

$\mathcal{L}_{\mathsf{Cat}}$

$$\begin{tikzcd}
T \arrow[dr, shift left=2] \arrow[dr] \arrow[dr, shift right=2] & I \arrow[d] & E_A \arrow[dl, shift left=1] \arrow[dl, shift right=1] \\
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\begin{tikzcd}
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- When we consider $\mathcal{L}_{\mathsf{Cat+E}}$-structures (with axioms), the notion of equivalence becomes isomorphism.

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$$\mathcal{L}_{\mathsf{Cat}} \qquad\qquad \mathcal{L}_{\mathsf{Cat+E}}$$

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▶ Different notions of equivalence are appropriate at different times.

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Equivalences for bicategories

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We can give different definitions of bicategory for each.

Double categories, formalized in UniMath

$$
\begin{tikzcd}
 & \text{Strict Dbl Setcat} \arrow[rr, "HV"] \arrow[dl, hook] \arrow[dd, hook] & & \text{2-Setcat} \arrow[dd, hook, dashed] \\
\text{Ps Dbl Setcat} \arrow[urrr, "V"] \arrow[drrr, "H"] \arrow[dr, hook] \arrow[dd, hook, dashed] & & & \\
 & \text{Weak Dbl Setcat} \arrow[rr, "HV"] \arrow[dd, hook, dashed] & & \text{Bisetcat} \arrow[dd, hook, dashed] \\
\text{Univ Ps Dbl Cat} \arrow[rr, "V"] \arrow[dr, hook, dashed] \arrow[drrr, dashed, "H"] \arrow[ddr, hook, bend right] & & \text{Univ 2-Cat} \arrow[ur, hook, dashed] \arrow[dr, hook, dashed] & \\
 & \text{Univ Weak Dbl Cat} \arrow[rr, "HV"] \arrow[d, hook] & & \text{Univ Bicat} \\
 & \text{Univ Dbl Bicat} \arrow[urr, "HV"] & &
\end{tikzcd}
$$

Thank you!