

(Towards a) Fuzzy type theory

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Outline

Introduction and motivation

Fuzzy propositional logic

Fuzzy type theory

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- ▶ To develop a language in which to model opinions
- ▶ To develop a type theory in which to verify fuzzy control systems
- ▶ To (begin to) generalize the correspondence between category theory and type theory to a correspondence with enriched category theory on one side
- ▶ To obtain another generalization of Martin-Löf type theory

What is an opinion?

- ▶ Logic of propositions
 - ▶ Model with complete lattices (posets with all co/limits)
 - ▶ Products (coproducts) represent conjunction (disjunction)
 - ▶ The terminal object \top (initial object \perp) represents the true (false) proposition
 - ▶ Write $P \leq Q$ to mean that P implies Q .
 - ▶ P holds when $\top \leq P$.

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- ▶ Logic of facts
 - ▶ Model with up-sets (slices) of lattices.
 - ▶ Given a lattice L of *propositions*, and a piece of *evidence* $e \in L$, e/L is the poset of propositions implied by e .
 - ▶ More generally, we can take a subcategory E of L .

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- ▶ Logic of opinions
 - ▶ Model with *fuzzy* lattices and *fuzzy* up-sets
 - ▶ Above, we answer “Is $P \leq Q$?” or “Does P hold?” with “yes” or “no”, i.e., “0” or “1”.
 - ▶ Now we answer “Is $P \leq Q$?” or “Does P hold?” with a value in an ordered monoid, for instance $[0, 1]$.

What is an opinion?

Proof irrelevant	Proof relevant
Propositions <ul style="list-style-type: none">• Posets• Categories enriched in $\{0, 1\}$	
Opinions <ul style="list-style-type: none">• Fuzzy posets• Categories enriched in $[0, 1]$	

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- ▶ Goal: develop the bottom-right box.

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- ▶ Opinion space was some real vector space.
- ▶ Modeling things as vectors plugs you in to a lot of computational tools,
- ▶ but it's akin to modeling propositional logic as a $\{0, 1\}$ -valued vector space.
- ▶ Want to capture more of the structure with a tailor-made algebraic notion.

Enriched categories

Booleans

- ▶ The natural ordering on the booleans $\mathbb{B} := \{0, 1\}$ forms a category.
- ▶ It has a monoidal structure given by multiplication.
- ▶ Thus, we can consider a \mathbb{B} -enriched category \mathcal{C} :
 - ▶ a set of objects $\text{ob}(\mathcal{C})$,
 - ▶ for each pair $x, y \in \text{ob}(\mathcal{C})$, an object $\text{hom}(x, y)$ of \mathbb{B} ,
 - ▶ for each $x \in \text{ob}(\mathcal{C})$, a point $1 \rightarrow \text{hom}(x, x)$
 - ▶ for each $x, y, z \in \text{ob}(\mathcal{C})$, a morphism $\circ : \text{hom}(x, y) \cdot \text{hom}(y, z) \rightarrow \text{hom}(x, z)$.
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We can interpret $\text{hom}(x, y)$ as indicating whether or not $x \leq y$.

Enriched categories

The interval

- ▶ The natural ordering on the interval $\mathbb{I} := [0, 1]$ forms a category.
- ▶ It has a monoidal structure given by multiplication.
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 - ▶ such that ...

We can interpret $\text{hom}(x, y)$ as indicating **to what extent** $x \leq y$.

Enriched categories

- ▶ In general, we can replace \mathbb{B} or \mathbb{I} with any monoidal category, but here we consider only monoidal categories which are posets, i.e., ordered monoids \mathbb{M} .
- ▶ Then, given an \mathbb{M} -enriched category \mathcal{C} (representing a space of opinions) we ask that it has the enriched (fuzzy) versions of all limits and colimits: all weighted limits and colimits.
- ▶ Then we consider a network of individuals, each with their own opinion space and opinion that they are communicating, and study dynamics.
 - ▶ Encode the network as a graph, and consider a sheaf over it, valued in the category of \mathbb{M} -enriched categories.

Weighted limits and colimits

- ▶ In a category, we can consider the product $A \times B$ of two objects, A , B
- ▶ But the concept of ‘weighted limits’ allows us to weight both A and B by sets α and β .
- ▶ The product with this weighting is then the product of α -many copies of A and β -many copies of B ($A^\alpha \times^\beta B$)
- ▶ In a \mathbb{M} -enriched category, to take a product of A and B , we take weights $\alpha, \beta \in M$.
- ▶ Then $A^\alpha \wedge^\beta B$ behaves like a conjunction of A scaled down by α and B scaled down by β .

Weighted meets and joins

Let:

- ▶ $S =$ “Alice likes strawberry ice cream.”
- ▶ $C =$ “Alice likes chocolate ice cream.”
- ▶ $B =$ “Alice likes chocolate ice cream better than strawberry ice cream.”
- ▶ $\alpha \in [0, 1]$

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Then we can consider:

- ▶ ${}^{\alpha}S =$ “Alice likes strawberry ice cream with intensity α .”
- ▶ $B^1 \wedge {}^{\alpha}S =$ “ B and ${}^{\alpha}S$ ”.

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Then we can consider:

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- ▶ $B^1 \wedge \alpha S =$ “ B and αS ”.

We can prove a ‘fuzzy modus ponens’:

- ▶ $(B^1 \wedge \alpha S \leq C) = \alpha$ and $(B^1 \wedge \alpha S \leq \alpha C) = 1$

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Fuzzy type theory (jww Shreya Arya, Greta Coraglia, Sean O'Connor, Hans Riess, Ana Tenório)

- ▶ In the last section, we fuzzified propositional logic by seeing it as a part of category theory, and fuzzifying the enrichment from \mathbb{B} to \mathbb{I} or \mathbb{M} .
- ▶ Now we fuzzify Martin-Löf type theory by a similar route.
- ▶ People might have multiple reasons for their opinions, so this seems appropriate.

Simple type theory

There is an equivalence of categories between simply typed λ -calculi and cartesian closed categories.

STLC	CCC
type A	object A
term $x : A \vdash b(x) : B$	morphism $b : A \rightarrow B$
conjunction $A \wedge B$	product $A \times B$
implication $A \Rightarrow B$	exponential B^A

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To fuzzify this, we consider on the right-hand side $\text{Set}(\mathbb{M})$ -enriched categories.

Fuzzy sets

$\text{Set}(\mathbb{M})$ is the category whose

- ▶ objects are pairs (X, ν) where X is a set and $\nu : X \rightarrow M$
- ▶ morphisms $(X, \nu) \rightarrow (Y, \mu)$ are functions $f : X \rightarrow Y$ such that $\nu(x) \leq \mu(fx)$ for all $x \in X$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \nu & \downarrow \mu \\ & & M \end{array}$$

It inherits a monoidal structure from the ones on Set and \mathbb{M} :

- ▶ $(X, \nu) \otimes (Y, \mu) := (X \times Y, \nu \cdot \mu)$
- ▶ The monoidal unit is $(*, 1)$.

Fuzzy categories

Definition

A $\text{Set}(\mathbb{M})$ -enriched category \mathcal{C} consists of

- ▶ a set of objects $\text{ob}(\mathcal{C})$,
 - ▶ for each pair $x, y \in \text{ob}(\mathcal{C})$, an object $\text{hom}(x, y)$ of $\text{Set}(\mathbb{M})$,
 - ▶ for each $x \in \text{ob}(\mathcal{C})$, a point $(1, *) \rightarrow \text{hom}(x, x)$
 - ▶ i.e., an element of $\text{hom}(x, x)$ with value 1
 - ▶ for each $x, y, z \in \text{ob}(\mathcal{C})$, a morphism $\circ : \text{hom}(x, y) \otimes \text{hom}(y, z) \rightarrow \text{hom}(x, z)$.
 - ▶ i.e., a function $\circ : \text{hom}(x, y) \times \text{hom}(y, z) \rightarrow \text{hom}(x, z)$ such that $|f||g| \leq |g \circ f|$
 - ▶ such that ...
- ▶ Now there can be multiple morphisms/reasons of a type/opinion, but each one comes with some intensity.

Dependent type theory

- ▶ We've talked about propositional logic and the simply typed λ -calculus, and their categorical interpretations.
- ▶ Our goal is actually dependent type theory.
 - ▶ Proof relevant first-order logic.
 - ▶ Types can be indexed by other types, just as predicates in first-order logic are indexed by sets.
 - ▶ In propositional logic, we have types/propositions A , in simply-types λ -calculus, we have terms/proofs $x : A \vdash b(x) : B$, and in dependent type theory we have dependent types $x : A \vdash B(x)$.

Display map categories

Definition

A *display map category* is a pair (\mathcal{C}, D) of a category \mathcal{C} and a class D of morphisms (called *display maps*) of \mathcal{C} such that

- ▶ \mathcal{C} has a terminal object $*$
 - ▶ every map $X \rightarrow *$ is a display map
 - ▶ D is stable under pullback
-
- ▶ The objects interpret types, the morphisms interpret terms, and the display maps interpret dependent types, and sections of display maps interpret dependent terms.
 - ▶ From a dependent type $x : B \vdash E(x)$, we can always form $\vdash \pi : \Sigma_{x:B} E(x) \rightarrow B$, and this is represented by the display maps.

Fuzzy display map categories

Definition

A *fuzzy display map category* is a pair (\mathcal{C}, D) of a $\mathbf{Set}(\mathbb{M})$ -enriched category \mathcal{C} and a class D of morphisms (called *fuzzy display maps*) of \mathcal{C} , each of which has value 1, such that

- ▶ \mathcal{C} has a terminal object $*$
- ▶ every map $X \rightarrow *$ is a display map
- ▶ D is stable under particular weighted pullbacks

Fuzzy terms

- ▶ The objects of a fuzzy display map category represent types (or contexts).
- ▶ The display maps $d : E \rightarrow B$ represent dependent types.
- ▶ In non-fuzzy display map categories, terms are represented as sections of display maps. Now our sections are fuzzy.

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- ▶ The objects of a fuzzy display map category represent types (or contexts).
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Definition

An α -fuzzy section of a fuzzy display map is a section with value at least α .

- ▶ These represent terms $x : B \vdash s :_{\alpha} E(x)$.

Substitution / weighted pullbacks

In the definition of *fuzzy display-map category*, we ask that the class of display maps is stable under particular weighted pullbacks.

$$\begin{array}{ccc} \bullet & \longrightarrow & E \\ \downarrow & \lrcorner & \downarrow d \\ A & \xrightarrow{f} & B \end{array}$$

- ▶ We choose the weight on A to be the singleton with value 1 and the weight on B to be the singleton with the value of f .
- ▶ Thus, the vertical maps have the same value (1), as do the horizontal maps.

Structural rules

$$\frac{}{\vdash_{\diamond} \text{ctx}} \text{ (C-Emp)}$$

$$\frac{\Gamma \vdash A \text{ Type}}{\vdash_{\Gamma, x:A} \text{ctx}} \text{ (C-Ext)}$$

$$\frac{\vdash_{\Gamma, x:A, \Delta} \text{ctx}}{\Gamma, x:A, \Delta \vdash x:1 A} \text{ (Var)}$$

$$\frac{\Gamma \vdash s:\alpha A}{\Gamma \vdash s:\beta A} \text{ for } \beta \leq \alpha \text{ (Cons)}$$

$$\frac{\Gamma, \Delta \vdash B \text{ Type} \quad \Gamma \vdash A \text{ Type}}{\Gamma, x:A, \Delta \vdash B \text{ Type}} \text{ (Weak}_{ty}\text{)}$$

$$\frac{\Gamma, \Delta \vdash b:\beta B \quad \Gamma \vdash A \text{ Type}}{\Gamma, x:A, \Delta \vdash b:\beta B} \text{ (Weak}_{tm}\text{)}$$

$$\frac{\Gamma, x:A, \Delta \vdash B \text{ Type} \quad \Gamma \vdash a:\alpha A}{\Gamma, \Delta[a/x] \vdash B[a/x] \text{ Type}} \text{ (Subst}_{ty}\text{)}$$

$$\frac{\Gamma, x:A, \Delta \vdash b:\beta B \quad \Gamma \vdash a:\alpha A}{\Gamma, \Delta[a/x] \vdash b[a/x]:\beta B[a/x]} \text{ (Subst}_{tm}\text{)}$$

Theorem

Fuzzy display map categories validate these rules.

Future work

Goals and questions

- ▶ Add type formers, like *weighted* conjunction
- ▶ Do we want to fuzzify other relations in type theory, like equality?
- ▶ Use this to study opinion dynamics

Thank you!