Coinductive control of inductive data types

Paige Randall North and Maximilien Péroux

Published in CALCO 2023
arXiv:2303.16793

18 July 2023

Outline

Overview

Theorem (N.-Péroux)

The category of algebras over an *accessible, lax symmetric monoidal* endofunctor on a *locally presentable, symmetric monoidal closed* category is enriched over the category of coalgebras of the same endofunctor.

Overview

Theorem (N.-Péroux)

The category of algebras over an *accessible, lax symmetric monoidal* endofunctor on a *locally presentable, symmetric monoidal closed* category is enriched over the category of coalgebras of the same endofunctor.

Examples

There are many examples, including polynomial endofunctors with extra structure.

Overview

Theorem (N.-Péroux)

The category of algebras over an *accessible, lax symmetric monoidal* endofunctor on a *locally presentable, symmetric monoidal closed* category is enriched over the category of coalgebras of the same endofunctor.

Examples

There are many examples, including polynomial endofunctors with extra structure.

Gain

Get more control over algebras

- Get more "initial algebras" (e.g. generalized W-types)

Natural numbers

Syntax

```
Inductive N : Type :=
| 0 :  N
| s :  N → N.
```

Natural numbers

Syntax

```
Inductive N : Type :=
| 0 :  N
| s :  N → N.
```

Categorical semantics

1. Consider the endofunctor $X \mapsto 1 + X$ on Set.
2. An *algebra* is a set X together with $\langle 0_X, s_X \rangle : 1 + X \to X$.
3. The initial algebra is $\mathbb{N}$.

Natural numbers

Syntax

```
Inductive N : Type :=
| 0 :  N
| s :  N → N.
```

Categorical semantics

1. Consider the endofunctor $X \mapsto 1 + X$ on Set.
2. An *algebra* is a set X together with $\langle 0_X, s_X \rangle : 1 + X \to X$.
3. The initial algebra is $\mathbb{N}$.

Coinductive data types and coalgebras

1. A *coalgebra* is a set X together with $X \to 1 + X$.
2. The terminal coalgebra is $\mathbb{N}^{\infty}$.

Lists

Syntax

```
Inductive list (A) : Type :=
| nil :  list (A)
| cons :  A → list(A) → list(A).
```

Lists

Syntax

```
Inductive list (A) : Type :=
| nil :  list (A)
| cons :  A → list(A) → list(A).
```

Categorical semantics

1. Consider the endofunctor $X \mapsto 1 + A \times X$ on Set.
2. An *algebra* is a set X with $\langle \mathrm{nil}_X, \mathrm{cons}_X \rangle : 1 + A \times X \to X$.
3. The initial algebra is $\mathbb{L}\mathrm{ist}(A)$.

Lists

Syntax

```
Inductive list (A) : Type :=
| nil :  list (A)
| cons :  A → list(A) → list(A).
```

Categorical semantics

1. Consider the endofunctor $X \mapsto 1 + A \times X$ on Set.
2. An *algebra* is a set X with $\langle \mathrm{nil}_X, \mathrm{cons}_X \rangle : 1 + A \times X \to X$.
3. The initial algebra is $\mathbb{L}\mathrm{ist}(A)$.

Coinductive data types and coalgebras

1. A *coalgebra* is a set X together with $X \to 1 + A \times X$.
2. The terminal coalgebra is $\mathbb{S}\mathrm{tream}(A)$.

Previous work on coalgebraic enrichment

Univeral measuring coalgebra (Wraith, Sweedler 1968)

For k-algebras A and B, there is a k-coalgebra $\underline{\mathrm{Alg}}(A, B)$

- which underlies an enrichment of k-algebras in k-coalgebras
- whose *set-like elements*[1] are in bijection with $\mathrm{Alg}(A, B)$.

Taking $B := k$, one gets the *dual* $\underline{\mathrm{Alg}}(A, k)$ of A.

Extensions

- Anel-Joyal 2013 (dg-algebras)
- Hyland-Franco-Vasilakopoulou 2017 (monoids)
- Vasilakopoulou 2019 ($\mathcal{V}$-categories)
- Péroux 2022 (∞-algebras of an ∞-operad)
- McDermott-Rivas-Uustalu 2022 (monads)

[1] those $c \in \underline{\mathrm{Alg}}(A, B)$ s.t. $\Delta c = c \otimes c$ and $\epsilon(c) = 1_A$

Enriched categories

Definition

An *enrichment* of a category $\mathcal{C}$ in a monoidal category $\mathcal{V}$ consists of

- a functor $\underline{\mathcal{C}}(-,-) : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{V}$
- a morphism $\mathbb{I} \to \underline{\mathcal{C}}(A, A)$ for each $A \in \mathrm{ob}\ \mathcal{C}$
- a morphism $\underline{\mathcal{C}}(A, B) \otimes \underline{\mathcal{C}}(B, C) \to \underline{\mathcal{C}}(A, C)$ for $A, B, C \in \mathrm{ob}\ \mathcal{C}$
- an isomorphism $\mathcal{V}(\mathbb{I}, \underline{\mathcal{C}}(A, B)) \cong \mathcal{C}(A, B)$ for $A, B \in \mathrm{ob}\ \mathcal{C}$.

such that ...

Remark

Monoidal *closed* means enriched in itself.

Measuring in general

Fix a *locally presentable, symmetric monoidal closed* category $\mathcal{C}$ and an *accessible, lax symmetric monoidal*endofunctor F.

Measuring

For algebras $(A, \alpha), (B, \beta)$ a *measure* $(A, \alpha) \to (B, \beta)$ is a coalgebra (C, χ) together with a morphism $\phi : C \to \underline{\mathcal{C}}(A, B)$ satisfying:

$$\begin{array}{ccccccc} & & FC & \xrightarrow{F(\phi)} & F(\underline{\mathcal{C}}(A,B)) & \longrightarrow & \underline{\mathcal{C}}(FA, FB) \\ & \overset{\chi}{\nearrow} & & & & & \downarrow \beta \\ C & & & & & & \\ & \underset{\phi}{\searrow} & & & & & \\ & & \underline{\mathcal{C}}(A,B) & & \xrightarrow{\alpha} & & \underline{\mathcal{C}}(FA, B) \end{array}$$

The *universal measure* $\underline{\mathrm{Alg}}(A, B)$ is the terminal one.

Measuring in general

Fix a *locally presentable, symmetric monoidal closed* category $\mathcal{C}$ and an *accessible, lax symmetric monoidal*endofunctor F.

Measuring

For algebras $(A, \alpha), (B, \beta)$ a *measure* $(A, \alpha) \to (B, \beta)$ is a coalgebra (C, χ) together with a morphism $\phi : C \to \underline{\mathcal{C}}(A, B)$ satisfying:

$$\begin{array}{ccccccc} & & FC & \xrightarrow{F(\phi)} & F(\underline{\mathcal{C}}(A, B)) & \longrightarrow & \underline{\mathcal{C}}(FA, FB) \\ & \overset{\chi}{\nearrow} & & & & & \downarrow \beta \\ C & & & & & & \\ & \underset{\phi}{\searrow} & & & & & \\ & & \underline{\mathcal{C}}(A, B) & & \xrightarrow{\qquad \alpha \qquad} & & \underline{\mathcal{C}}(FA, B) \end{array}$$

The *universal measure* $\underline{\mathrm{Alg}}(A, B)$ is the terminal one.

Theorem (N.-Péroux)

The universal measure $\underline{\mathrm{Alg}}(A, B)$ always exists, and these are the hom-coalgebras of an enrichment of $\mathrm{Alg}(F)$ in $\mathrm{CoAlg}(F)$.

Measuring for the natural numbers

Measuring

For algebras A, B, a *measure* $A \to B$ is a coalgebra C together with a function $C \to A \to B$ such that

- $f_c(0_A) = 0_B$ for all $c \in C$;
- $f_c(a+1) = 0_B$ for all $[\![c]\!] = 0$ and for all $a \in A$;
- $f_c(a+1) = f_{c-1}(a) + 1$ for $[\![c]\!] \geqslant 1$ and for all $a \in A$.

The *universal measure* $\underline{\mathrm{Alg}}(A, B)$ is the terminal measure $A \to B$.

Measuring for the natural numbers

Measuring

For algebras A, B, a *measure* $A \to B$ is a coalgebra C together with a function $C \to A \to B$ such that

- $f_c(0_A) = 0_B$ for all $c \in C$;
- $f_c(a+1) = 0_B$ for all $[\![c]\!] = 0$ and for all $a \in A$;
- $f_c(a+1) = f_{c-1}(a) + 1$ for $[\![c]\!] \geqslant 1$ and for all $a \in A$.

The *universal measure* $\underline{\mathrm{Alg}}(A, B)$ is the terminal measure $A \to B$.

What is this?

Set-like elements in general

Definition

The *set-like elements* are

$$\mathbb{1} \to \underline{\mathrm{Alg}}(A, B) \qquad \text{in } \mathrm{CoAlg}(F)$$

i.e., elements of $\mathrm{Alg}(A, B)$.

Set-like elements in general

Definition

The *set-like elements* are

$$\mathbb{1} \to \underline{\mathsf{Alg}}(A, B) \qquad \text{in } \mathsf{CoAlg}(F)$$

i.e., elements of $\mathsf{Alg}(A, B)$.

That is

- The *points* of $\mathsf{Alg}(A, B)$ are total algebra homomorphisms $A \to B$.

Set-like elements in general

Definition

The *set-like elements* are

$$\mathbb{I} \to \underline{\mathrm{Alg}}(A, B) \qquad \text{in } \mathrm{CoAlg}(F)$$

i.e., elements of $\mathrm{Alg}(A, B)$.

That is

- The *points* of $\mathrm{Alg}(A, B)$ are total algebra homomorphisms $A \to B$.
- If we're considering $(\mathrm{Set}, \times, *)$, the underlying set of $\mathbb{I}$ is $*$, so these are 'special' elements of the underlying set of $\mathrm{Alg}(A, B)$.

Set-like elements for the natural numbers

Set-like elements

The *set-like elements* are

$$\mathbb{1} \to \underline{\mathsf{Alg}}(A, B)$$

Set-like elements for the natural numbers

Set-like elements

The *set-like elements* are

$$\mathbb{I} \to \underline{\mathsf{Alg}}(A, B)$$

where $\mathbb{I}$ has underlying set $\{*\}$ such that $[\![*]\!] = \infty$

Set-like elements for the natural numbers

Set-like elements

The *set-like elements* are

$$\mathbb{I} \to \underline{\mathsf{Alg}}(A, B)$$

where $\mathbb{I}$ has underlying set $\{*\}$ such that $[\![*]\!] = \infty$
so $\mathbb{I} \to \underline{\mathsf{Alg}}(A, B)$ is an element $* \in \underline{\mathsf{Alg}}(A, B)$ s.t. $[\![*]\!] = \infty$

Set-like elements for the natural numbers

Set-like elements

The *set-like elements* are

$$\mathbb{I} \to \underline{\mathrm{Alg}}(A, B)$$

where $\mathbb{I}$ has underlying set $\{*\}$ such that $[\![*]\!] = \infty$
so $\mathbb{I} \to \underline{\mathrm{Alg}}(A, B)$ is an element $* \in \underline{\mathrm{Alg}}(A, B)$ s.t. $[\![*]\!] = \infty$
so f_* is a total algebra homomorphism

Measuring

...

- $f_c(0_A) = 0_B$ for all $c \in C$;

  ...

- $f_c(a + 1) = f_{c-1}(a) + 1$ for $[\![c]\!] \geqslant 1$ and for all $a \in A$.

Set-like elements for the natural numbers

Set-like elements

The *set-like elements* are

$$\mathbb{I} \to \underline{\mathrm{Alg}}(A, B)$$

where $\mathbb{I}$ has underlying set $\{*\}$ such that $[\![*]\!] = \infty$
so $\mathbb{I} \to \underline{\mathrm{Alg}}(A, B)$ is an element $* \in \underline{\mathrm{Alg}}(A, B)$ s.t. $[\![*]\!] = \infty$
so f_* is a total algebra homomorphism

Measuring

...

- $f_*(0_A) = 0_B$;

  ...

- $f_*(a + 1) = f_*(a) + 1$ for all $a \in A$.

Set-like elements for the natural numbers

Set-like elements

The *set-like elements* are

$$\mathbb{1} \to \underline{\mathrm{Alg}}(A, B)$$

where $\mathbb{1}$ has underlying set $\{*\}$ such that $[\![*]\!] = \infty$
so $\mathbb{1} \to \underline{\mathrm{Alg}}(A, B)$ is an element $* \in \underline{\mathrm{Alg}}(A, B)$ s.t. $[\![*]\!] = \infty$
so f_* is a total algebra homomorphism
that is, an element of $\mathrm{Alg}(A, B)$.

Measuring

...

- $f_*(0_A) = 0_B$;

  ...
- $f_*(a + 1) = f_*(a) + 1$ for all $a \in A$.

Set-like elements for the natural numbers

Set-like elements

The *set-like elements* are

$$\mathbb{I} \to \underline{\mathrm{Alg}}(A, B)$$

where $\mathbb{I}$ has underlying set $\{*\}$ such that $[\![*]\!] = \infty$
so $\mathbb{I} \to \underline{\mathrm{Alg}}(A, B)$ is an element $* \in \underline{\mathrm{Alg}}(A, B)$ s.t. $[\![*]\!] = \infty$
so f_* is a total algebra homomorphism
that is, an element of $\mathrm{Alg}(A, B)$.

Example

$$\mathrm{Alg}(\mathbb{N}, A) \cong *$$
$$\underline{\mathrm{Alg}}(\mathbb{N}, A) \cong \mathbb{N}^{\infty}$$

What are the non-set-like elements?

Example

$$\underline{\mathsf{Alg}}(\mathbb{N}, A) \cong \mathbb{N}^{\infty}$$

What are the non-set-like elements?

Example

$$\underline{\text{Alg}}(\mathbb{N}, A) \cong \mathbb{N}^{\infty}$$

So denote the elements of $\underline{\text{Alg}}(\mathbb{N}, A)$ by

- f_0
- f_1

  ...
- f_∞

Measuring

...

- $f_0(a+1) = 0_B$ and for all $a \in A$

What are the non-set-like elements?

Example

$$\underline{\mathrm{Alg}}(\mathbb{N}, A) \cong \mathbb{N}^{\infty}$$

So denote the elements of $\underline{\mathrm{Alg}}(\mathbb{N}, A)$ by

- $f_0(n) = 0_A$
- f_1

  $\dots$
- f_∞

Measuring

...

- $f_0(a+1) = 0_B$ and for all $a \in A$

What are the non-set-like elements?

Example

$$\underline{\mathrm{Alg}}(\mathbb{N}, A) \cong \mathbb{N}^{\infty}$$

So denote the elements of $\underline{\mathrm{Alg}}(\mathbb{N}, A)$ by

- $f_0(n) = 0_A$
- f_1

  ...
- f_∞

Measuring

...

- $f_1(0_A) = 0_B$
- $f_1(a+1) = f_0(a) + 1$ for all $a \in A$

What are the non-set-like elements?

Example

$$\underline{\mathrm{Alg}}(\mathbb{N}, A) \cong \mathbb{N}^{\infty}$$

So denote the elements of $\underline{\mathrm{Alg}}(\mathbb{N}, A)$ by

- $f_0(n) = 0_A$
- $f_1(0) = 0_A$; $f_1(sn) = 1_A$

  ...
- f_∞

Measuring

...

- $f_1(0_A) = 0_B$
- $f_1(a + 1) = f_0(a) + 1$ for all $a \in A$

What are the non-set-like elements?

Example

$$\underline{\mathrm{Alg}}(\mathbb{N}, A) \cong \mathbb{N}^{\infty}$$

So denote the elements of $\underline{\mathrm{Alg}}(\mathbb{N}, A)$ by

- $f_0(n) = 0_A$
- $f_1(0) = 0_A$; $f_1(sn) = 1_A$

  ...
- f_∞

Measuring

...

- $f_\infty(0) = 0_B$
- $f_\infty(a+1) = f_\infty(a) + 1$

What are the non-set-like elements?

Example

$$\underline{\mathrm{Alg}}(\mathbb{N}, A) \cong \mathbb{N}^{\infty}$$

So denote the elements of $\underline{\mathrm{Alg}}(\mathbb{N}, A)$ by

- $f_0(n) = 0_A$
- $f_1(0) = 0_A$; $f_1(sn) = 1_A$

  ...
- $f_\infty(n) = n_A$

Measuring

...

- $f_\infty(0) = 0_B$
- $f_\infty(a+1) = f_\infty(a) + 1$

What are the non-set-like elements?

Example

$$\underline{\mathsf{Alg}}(\mathbb{N}, A) \cong \mathbb{N}^{\infty}$$

So denote the elements of $\underline{\mathsf{Alg}}(\mathbb{N}, A)$ by

- $f_0(n) = 0_A$
- $f_1(0) = 0_A; f_1(sn) = 1_A$

  ...
- $f_\infty(n) = n_A$

Definition

So we call elements of the underlying of $\underline{\mathsf{Alg}}(A, B)$ *n-partial algebra homomorphisms*.

What are the non-set-like elements?

- Let $\mathbb{n}$ denote the quotient of $\mathbb{N}$ by $m = n$ for all $m \geqslant n$.
- Let $\mathbb{n}^\circ$ denote the subobject of $\mathbb{N}^\infty$ consisting of $\{0, \ldots, n\}$.

Example

$$\mathrm{Alg}(\mathbb{n}, A) \cong \begin{cases} * & \text{if } n_A = m_A \text{ for all } m \geqslant n; \\ \varnothing & \text{otherwise.} \end{cases}$$

What are the non-set-like elements?

- Let $\mathbb{n}$ denote the quotient of $\mathbb{N}$ by $m = n$ for all $m \geqslant n$.
- Let $\mathbb{n}^\circ$ denote the subobject of $\mathbb{N}^\infty$ consisting of $\{0, \ldots, n\}$.

Example

$$\mathrm{Alg}(\mathbb{n}, A) \cong \begin{cases} * & \text{if } n_A = m_A \text{ for all } m \geqslant n; \\ \varnothing & \text{otherwise.} \end{cases}$$

$$\underline{\mathrm{Alg}}(\mathbb{n}, A) \cong \begin{cases} \mathbb{N}^\infty & \text{if } n_A = m_A \text{ for all } m \geqslant n; \\ \mathbb{n}^\circ & \text{otherwise.} \end{cases}$$

- So there is at least always an n-partial homomorphism out of n (which is unique).

What can we do with this?

Generalize W-types, i.e., initial algebras.

C-initial objects

For a coalgebra C, a *C-initial algebra* is an algebra A such that for all other algebras B there is a unique

$$C \to \underline{\mathsf{Alg}}(A, B).$$

Initial object

An initial object in a category $\mathcal{C}$ is an object A such that for all other algebras B there is a unique

$$* \to \mathcal{C}(A, B).$$

C-initial objects for the natural numbers

Examples

For the natural-numbers endofunctor:

- $\mathbb{N}$ is the $\mathbb{I}$-*initial algebra*
- $\mathbb{N}$ is the $\mathbb{N}^\infty$-*initial algebra*

C-initial objects for the natural numbers

Examples

For the natural-numbers endofunctor:

- $\mathbb{N}$ is the $\mathbb{I}$-*initial algebra*
- $\mathbb{N}$ is the $\mathbb{N}^{\infty}$-*initial algebra*

- $\mathbb{I}$- (or $\mathbb{N}^{\infty}$-) initial means initial with respect to total algebra homomorphisms

Theorem

$\mathbb{n}$ is the $\mathbb{n}^{\circ}$-*initial algebra*

- $\mathbb{n}^{\circ}$-initial means initial with respect to partial algebra homomorphisms

Examples

(Endofunctors on a locally presentable symmetric monoidal category)

- (id) The identity endofunctor
- (A) The constant endofunctor at fixed commutative monoid A
- (GF) The composition of two instances
- ($F \otimes G$) The tensor of two instances ($\mathcal{C}$ closed)
- ($F + G$) The coproduct of an instance F and an 'F-module' G
- (id^A) The exponential id^A at object A ($\mathcal{C}$ cartesian closed)
- (W-types) The polynomial endofunctor associated to a morphism $f : X \to Y$, given a commutative monoid structure on Y and an oplax symmetric monoidal structure on the preimage functor $f^{-1} : C \to \mathrm{Set}$ ($\mathcal{C} = \mathrm{Set}$)
- (d.e.s.) A discrete equational system (monoidal structure on $\mathcal{C}$ is cocartesian, $\mathcal{C}$ has binary products that preserve filtered colimits)

Summary

We have

- that algebras are enriched in coalgebras (under certain hypotheses)
- an interpretation as notion of partial algebra homomorphism (especially in the case N)
- many examples
- a more refined notion of initial algebra

Future work

- Work out more of the examples in detail
- Understand C-initial algebras in more examples and in general
- Understand if this extra structure is useful for programming languages
- Understand if there is a connection with domain theory

Thank you!