Formalizing the algebraic small object argument in UniMath

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Outline

Introduction and motivation

Our approach

Results

- We formalized (the relevant part of) Garner's small object argument (SOA) in Coq UniMath

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- The SOA is a fundamental tool in modern homotopy theory
  - Used to construct most *model categories*
  - Used for most models of HoTT/UF
- So our goal is to build the tools to formalize models of HoTT/UF
  - e.g. Voevodsky's model in simplicial sets

Homotopy type theory / univalent foundations

HoTT/UF arises from the observation that in the denotational (categorical) semantics of Martin-Löf Type Theory:[1]

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Equality type	$\leftrightsquigarrow$	Path space
Dependent types	$\leftrightsquigarrow$	Fibrations (right lifting property)

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[2]Voevodsky

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- HoTT $=$ UF $+$ higher inductive types

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- UF = MLTT + Univalence axiom[2]
- HoTT = UF + higher inductive types
- Coq UniMath = Coq - Prop + Univalence axiom

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$$X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots$$

where $X_{-1} := \emptyset$ and X_{n+1} is obtained from X_n by gluing disks D^m along their boundaries S^{m-1} to X_n.

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SOA 2: approximation by CW complexes

Given a space X, can approximate by a *relative* CW complex.

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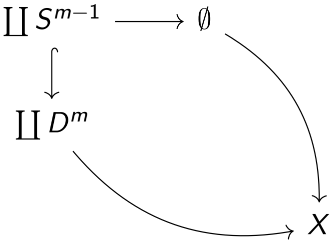

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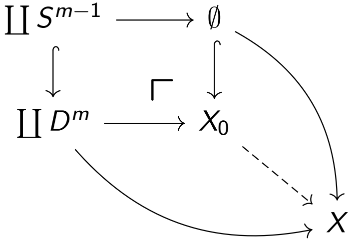

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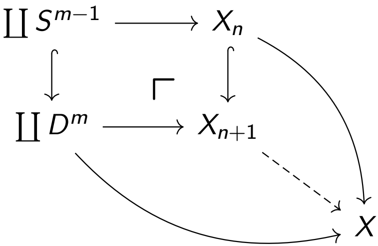

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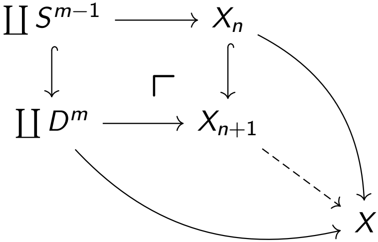
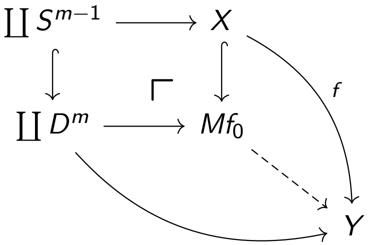


Get a sequence

$$\emptyset \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_\infty \to X.$$

SOA 2: approximation by CW complexes

Given a *map* $f : X \to Y$, can approximate by a *relative* CW complex.



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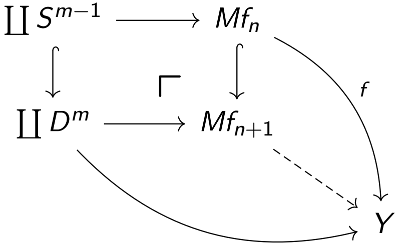

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SOA 3: The classical statement

The small object argument

Quillen's small object argument gives conditions that ensure that

- given a class of maps $\mathcal{I}$ in a category $\mathcal{C}$
  - e.g. $\{S^{n-1} \hookrightarrow D^n\}_n$ in topological spaces
- any map $f : X \to Y$ factorizes as $X \to Mf_\infty \to Y$ where $Mf_\infty \to Y$ has the *right lifting property* with respect to $\mathcal{I}$.

Use (with some niceness conditions)

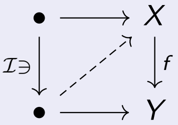
Model Martin-Löf's identity type by factoring the diagonal

$$X \longrightarrow M\Delta_\infty \longrightarrow X \times X \qquad (\Delta : X \to X \times X)$$

Lifting properties

Right lifting property

A map $f : X \to Y$ has the *right lifting property* against $\mathcal{I}$ if for all (solid) commutative squares, there exists a (dashed) lift making the diagram commute.



Use (with omitted details)

Transport can be characterized as a right lifting property.

Constructivity

- The “exists” above is classical existence.
  - Leads to problems without axiom of choice.
- Instead, we turn to *algebraic* versions of these concepts, where the lift is given as *algebraic structure*

Algebraic small object argument

Garner's small object argument

Garner's small object argument gives conditions that ensure that

- given a small category of maps $\mathcal{I}$ in a category $\mathcal{C}$
- any map $f : X \to Y$ factorizes as $X \xrightarrow{Lf} Mf_\infty \xrightarrow{Rf} Y$ where L is a comonad and R is a monad (this ensures the lifting condition).
- Moreover, the construction 'converges': while Quillen's SOA adds the same cells over and over again, Garner's adds them only as needed.

Our result

We formalize Garner's SOA.

Outline

UniMath

- We use UniMath because our experience convinces us that it's the right setting in which to formalize category theory.
- However, the univalence axiom was not needed.
- Thus, we use MLTT (Coq - Prop) with the following concepts from HoTT/UF:
  - homotopy levels (propositions, sets)
  - propositional truncation
  - displayed categories

Categories

- We don't use the univalence axiom or univalent categories.
- We use the *categories* of the UniMath library (called precategories in the HoTT book).

Category

A *category* consists of a type $\mathtt{ob} : \mathtt{UU}$ and a dependent type $\mathtt{mor} : \mathtt{ob} \to \mathtt{ob} \to \mathtt{hSet}$ together with identities, composition, and axioms.

- These can be assumed to be
  - setcategories: $\mathtt{ob}$ is a (homotopy) set
  - univalent:[3] the morphism $(x = y) \to (x \cong y)$ is an equivalence for $x, y : \mathtt{ob}$, making $\mathtt{ob}$ a (homotopy) groupoid

[3] Ahrens-Kapulkin-Shulman

Displayed categories[4]

- In HoTT/UF, the equality type behaves weakly.
  - If a category is univalent, the equality type captures isomorphism.
- To define *factorization*, classically we ask for a section of the functor $\circ : \mathcal{C}^{\to\to} \to \mathcal{C}^{\to}$.
- To do this in UniMath, we encode $\mathcal{C}^{\to\to}$ as a *displayed category* over $\mathcal{C}^{\to}$ (analogous to dependent type).
  - Over each $f \in \mathcal{C}^{\to}$: the type of factorizations of f.
  - Then a section consists of a (functorial) factorization of each f (analogous to dependent term).
- I.e. describing a section using equality is too weak, so we use displayed categories to access strict equality

[4] Ahrens-Lumsdaine

Monoidal categories

- We also encode the categories of (left/right) natural weak factorization systems on a category $\mathcal{C}$ as displayed monoidal categories over functorial factorizations.
- Where Garner uses *strict* monoidal categories (i.e. certain axioms hold up to classical equality), we generalize to monoidal categories.
  - Again, because we do not have access to classical equality

Limitations and future work

- We only formalized the SOA where the transfinite composition – i.e., the sequence

$$X \subseteq Mf_0 \subseteq \cdots$$

  – has length ω.
  - Would like to generalize this to more general ordinals.
- Continue this program of formalizing the semantics of HoTT in HoTT.

Thank you!