

# Weak factorization systems as models of dependent type theory

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# Overview: preliminaries

## Problem

To identify those weak factorization systems which harbor a model of the  $\Sigma$ ,  $\Pi$ , and  $\text{Id}$  types of dependent type theory.

## Formulation

Use the theory of *display map categories* to formulate our problem.

# Display map categories and weak factorization systems

## Definition of *display map category*

A category  $\mathcal{C}$  with a terminal object and a class of *display maps*  $\mathcal{D}$  satisfying closure properties (has all isos & maps to  $*$ ; all PBs of display maps exist & are display maps).

- ▶ The simplest categorical framework for modeling dependent type theory
- ▶ The closest to the theory of weak factorization systems

## Definition of *weak factorization system*

A category  $\mathcal{C}$  with two classes of maps  $(\mathcal{L}, \mathcal{R})$  such that every map of  $\mathcal{C}$  factors into an  $\mathcal{L}$ -map followed by an  $\mathcal{R}$ -map,  $\mathcal{L}$  are the maps with the left lifting property against  $\mathcal{R}$ , and vice versa.

## Overview: results

### Theorem 1: reducing the problem.

Let  $\mathcal{C}$  be a Cauchy complete category with a class of display maps  $\mathcal{D}$  which model  $\Sigma$  and Id types. Then:

1.  $\overline{\mathcal{D}}$  (the retract closure of  $\mathcal{D}$ ) is itself a class of display maps which models  $\Sigma$  and Id types.
2. And if  $\mathcal{D}$  models  $\Pi$  types, then  $\overline{\mathcal{D}}$  models  $\Pi$  types.

$(\square\mathcal{D}, \overline{\mathcal{D}})$  is a weak factorization system (Gambino-Garner, 2008).

Thus:

- ▶ Every model  $\mathcal{D}$  of  $\Sigma$ , Id ( $\Pi$ ) types 'lives inside' another model  $\overline{\mathcal{D}}$  which is the right class of a weak factorization system.
- ▶ To know whether a WFS  $(\mathcal{L}, \mathcal{R})$  'harbors' a model of  $\Sigma$ , Id ( $\Pi$ ) types, we need only determine whether  $\mathcal{R}$  itself is a model.

## Overview: results

### Theorem 2: the characterization.

Let  $\mathcal{C}$  be a finitely complete category with a weak factorization system  $(\mathcal{L}, \mathcal{R})$ . Then the following are equivalent:

1.  $\mathcal{R}$  is a class of display maps which models  $\Sigma$  and  $\text{Id}$  types.
2. Every map to the terminal object is in  $\mathcal{R}$  and  $\mathcal{L}$  is stable under pullback along  $\mathcal{R}$ .
3.  $(\mathcal{L}, \mathcal{R})$  is generated by a Moore relation structure.

And if the above are true and  $\mathcal{C}$  is locally cartesian closed, then  $\mathcal{R}$  models  $\Pi$  types.

- ▶ Moore relation structure: algebraic data which generate the WFS

## $\Sigma$ and $\Pi$ types

### Definition of $\Sigma$ types

A DMC  $(\mathcal{C}, \mathcal{D})$  models  $\Sigma$  types when  $\mathcal{D}$  is closed under composition.

### Definition of $\Pi$ types

A DMC  $(\mathcal{C}, \mathcal{D})$  models  $\Pi$  types when for all

$$W \xrightarrow{g} X \xrightarrow{f} Y \in \mathcal{D}$$

there is a display map representing

$$\mathrm{hom}_{\mathcal{C}/X}(f^* -, g) : (\mathcal{C}/Y)^{op} \rightarrow \mathit{Set}.$$

# Id types

## Definition of Id types

A DMC  $(\mathcal{C}, \mathcal{D})$  models Id types *on objects* when for every object  $Y$  of  $\mathcal{C}$ , there is a factorization of the diagonal

$$Y \xrightarrow{r} \text{Id}(Y) \xrightarrow{\epsilon_0 \times \epsilon_1} Y \times Y$$

such that  $\epsilon_0 \times \epsilon_1$  is in  $\mathcal{D}$  and every pullback of  $r$  has the left lifting property against  $\mathcal{D}$  (or: is in  $\square\mathcal{D}$ ).

The diagram illustrates the factorization of the diagonal for an object  $X$ . It consists of two triangles and a connecting arrow. On the left, a triangle has vertices  $X$  (top-left),  $X \times_Y \text{Id}(Y)$  (top-right), and  $X$  (bottom). The top edge is labeled  $1 \times rf$ , the right edge is labeled  $\pi$ , and the bottom edge is a double line. On the right, a triangle has vertices  $Y$  (top-left),  $\text{Id}(Y)$  (top-right), and  $Y$  (bottom). The top edge is labeled  $r$ , the right edge is labeled  $\epsilon_i$ , and the bottom edge is a double line. A horizontal arrow labeled  $f$  connects the bottom vertex  $X$  of the left triangle to the bottom vertex  $Y$  of the right triangle.

## The factorization

- ▶  $\Sigma$  and  $\text{Id}$  types produce a factorization of any map  $f : X \rightarrow Y$

$$X \xrightarrow{1 \times rf} X \times_Y \text{Id}(Y) \xrightarrow{\epsilon_1 \pi} Y$$

where the left map is in  $\square\mathcal{D}$  and the right map is in  $\mathcal{D}$ .

- ▶ This generates a weak factorization system  $(\square\mathcal{D}, \overline{\mathcal{D}})$ .
- ▶  $\overline{\mathcal{D}}$  is the retract closure of  $\mathcal{D}$

A commutative square diagram with four vertices. The top and bottom edges are horizontal arrows pointing to the right. The left and right edges are vertical arrows pointing downwards. The left vertical arrow is labeled 'r', the right vertical arrow is labeled 'r', and the central vertical arrow is labeled 'd'. The diagram represents the relationship between the square of the factorization and its retract closure.



## Cauchy complete categories.

### Definition: Cauchy complete category

A category for which any retract of a representable presheaf is itself representable.

- ▶ DMC,  $\Sigma$ , and  $\Pi$  types can be phrased in terms of the existence of representable functors built out of display maps.
- ▶ So can deduce existence of those built from  $\overline{\mathcal{D}}$  from the existence of those built from  $\mathcal{D}$ .

### Theorem 1

$(\mathcal{C}, \mathcal{D})$  a CC DMC modeling  $\Sigma$  and Id (and  $\Pi$ ) types. Then  $(\mathcal{C}, \overline{\mathcal{D}})$  is a DMC modeling  $\Sigma$  and Id (and  $\Pi$ ) types.

# The characterization

$\mathcal{C}$  finitely complete category.

- ▶  $\mathcal{W} :=$  category of WFSs  $(\mathcal{L}, \mathcal{R})$  on  $\mathcal{C}$  where every map to the terminal object is in  $\mathcal{R}$  and  $\mathcal{L}$  is stable under pullback along  $\mathcal{R}$ .
- ▶  $\mathcal{I} :=$  category of Id-types-on-objects in  $\mathcal{C}$ .

There are functors:

$$F : \mathcal{W} \rightleftarrows \mathcal{I} : G$$

- ▶  $F$  takes a WFS to its factorization of the diagonal maps
- ▶  $G$  builds a WFS from an Id-types-on-objects as we did above

Using Moore relation structures, we find:

## Theorem 2

$F$  and  $G$  form an equivalence of categories.

Thank you!