Weak factorization systems as models of dependent type theory

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Overview: preliminaries

Problem

To identify those weak factorization systems which harbor a model of the Σ, Π, and Id types of dependent type theory.

Formulation

Use the theory of *display map categories* to formulate our problem.

Display map categories and weak factorization systems

Definition of *display map category*

A category $\mathcal{C}$ with a terminal object and a class of *display maps* $\mathcal{D}$ satisfying closure properties (has all isos & maps to $*$; all PBs of display maps exist & are display maps).

- The simplest categorical framework for modeling dependent type theory
- The closest to the theory of weak factorization systems

Definition of *weak factorization system*

A category $\mathcal{C}$ with two classes of maps $(\mathcal{L}, \mathcal{R})$ such that every map of $\mathcal{C}$ factors into an $\mathcal{L}$-map followed by an $\mathcal{R}$-map, $\mathcal{L}$ are the maps with the left lifting property against $\mathcal{R}$, and vice versa.

Overview: results

Theorem 1: reducing the problem.

Let $\mathcal{C}$ be a Cauchy complete category with a class of display maps $\mathcal{D}$ which model Σ and Id types. Then:

1. $\overline{\mathcal{D}}$ (the retract closure of $\mathcal{D}$) is itself a class of display maps which models Σ and Id types.
2. And if $\mathcal{D}$ models Π types, then $\overline{\mathcal{D}}$ models Π types.

$({}^{\boxslash}\mathcal{D}, \overline{\mathcal{D}})$ is a weak factorization system (Gambino-Garner, 2008). Thus:

- Every model $\mathcal{D}$ of Σ, Id (Π) types 'lives inside' another model $\overline{\mathcal{D}}$ which is the right class of a weak factorization system.
- To know whether a WFS $(\mathcal{L}, \mathcal{R})$ 'harbors' a model of Σ, Id (Π) types, we need only determine whether $\mathcal{R}$ itself is a model.

Overview: results

Theorem 2: the characterization.

Let $\mathcal{C}$ be a finitely complete category with a weak factorization system $(\mathcal{L}, \mathcal{R})$. Then the following are equivalent:

1. $\mathcal{R}$ is a class of display maps which models Σ and Id types.
2. Every map to the terminal object is in $\mathcal{R}$ and $\mathcal{L}$ is stable under pullback along $\mathcal{R}$.
3. $(\mathcal{L}, \mathcal{R})$ is generated by a Moore relation structure.

And if the above are true and $\mathcal{C}$ is locally cartesian closed, then $\mathcal{R}$ models Π types.

- Moore relation structure: algebraic data which generate the WFS

Σ and Π types

Definition of Σ types

A DMC $(\mathcal{C}, \mathcal{D})$ models Σ types when $\mathcal{D}$ is closed under composition.

Definition of Π types

A DMC $(\mathcal{C}, \mathcal{D})$ models Π types when for all

$$W \xrightarrow{g} X \xrightarrow{f} Y \quad \in \mathcal{D}$$

there is a display map representing

$$\hom_{\mathcal{C}/X}(f^* -, g) : (\mathcal{C}/Y)^{op} \to Set.$$

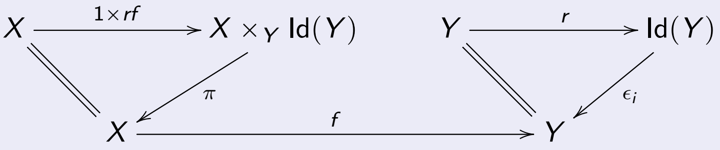
Id types

Definition of Id types

A DMC $(\mathcal{C}, \mathcal{D})$ models Id types *on objects* when for every object Y of $\mathcal{C}$, there is a factorization of the diagonal

$$Y \xrightarrow{r} \mathrm{Id}(Y) \xrightarrow{\epsilon_0 \times \epsilon_1} Y \times Y$$

such that $\epsilon_0 \times \epsilon_1$ is in $\mathcal{D}$ and every pullback of r has the left lifting property against $\mathcal{D}$ (or: is in ${}^{\boxslash}\mathcal{D}$).



The factorization

- Σ and Id types produce a factorization of any map $f : X \to Y$

$$X \xrightarrow{1 \times rf} X \times_Y \mathrm{Id}(Y) \xrightarrow{\epsilon_1 \pi} Y$$

  where the left map is in ${}^{\boxslash}\mathcal{D}$ and the right map is in $\mathcal{D}$.
- This generates a weak factorization system $({}^{\boxslash}\mathcal{D}, \overline{\mathcal{D}})$.
- $\overline{\mathcal{D}}$ is the retract closure of $\mathcal{D}$

$$\begin{array}{ccccc} & \longrightarrow & & \longrightarrow & \\ \downarrow r & & \downarrow d & & \downarrow r \\ & \longrightarrow & & \longrightarrow & \end{array}$$

Cauchy complete categories.

Definition: Cauchy complete category

A category for which any retract of a representable presheaf is itself representable.

- DMC, Σ, and Π types can be phrased in terms of the existence of representable functors built out of display maps.
- So can deduce existence of those built from $\overline{\mathcal{D}}$ from the existence of those built from $\mathcal{D}$.

Theorem 1

$(\mathcal{C}, \mathcal{D})$ a CC DMC modeling Σ and Id (and Π) types. Then $(\mathcal{C}, \overline{\mathcal{D}})$ is a DMC modeling Σ and Id (and Π) types.

The characterization

$\mathcal{C}$ finitely complete category.

- $\mathcal{W}$:= category of WFSs $(\mathcal{L}, \mathcal{R})$ on $\mathcal{C}$ where every map to the terminal object is in $\mathcal{R}$ and $\mathcal{L}$ is stable under pullback along $\mathcal{R}$.
- $\mathcal{I}$:= category of Id-types-on-objects in $\mathcal{C}$.

There are functors:

$$F : \mathcal{W} \leftrightarrows \mathcal{I} : G$$

- F takes a WFS to its factorization of the diagonal maps
- G builds a WFS from an Id-types-on-objects as we did above

Using Moore relation structures, we find:

Theorem 2

F and G form an equivalence of categories.

Thank you!