The Equivalence Principle and Univalent Foundations

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Outline

The equivalence principle

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Reasoning in mathematics should be **invariant under** the appropriate notion of **equivalence**.

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Reasoning in mathematics should be **invariant under** the appropriate notion of **equivalence**.

Notion of equivalence depends on the objects under consideration:

- **equal** numbers, functions,. . .
- **isomorphic** sets, groups, rings,. . .
- **equivalent** categories
- **biequivalent** bicategories
- . . .

Non-examples: statements violating equivalence principle

We can easily **violate** this principle:

Exercise

Find a statement about sets that is not invariant under isomorphism:

$$\{\emptyset, \{\emptyset\}\} \cong \{\emptyset, \{\{\emptyset\}\}\}$$

Exercise

Find a statement about categories that is not invariant under equivalence:

$$\begin{tikzcd} \bullet \arrow[r, bend left] & \bullet \arrow[l, bend left] \end{tikzcd} \quad \simeq \quad \bullet$$

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$\{\emptyset\} \in X$

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Find a statement about categories that is not invariant under equivalence:

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$\mathscr{C}$ has exactly 1 object.

A language for invariant properties

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> *"The basic character of the Principle of Isomorphism is that of a* ***constraint on the language*** *of Abstract Mathematics; a welcome one, since it provides for the separation of sense from nonsense."*

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Goal

To have a **syntactic criterion** for properties and constructions that are invariant under equivalence

How to break the equivalence principle for categories. . .

- Recall: the statement

  The category $\mathscr{C}$ has exactly one object.

  is not invariant under equivalence of categories.
- In general, referring to **equality of objects** breaks invariance, but. . .

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- even the **definition** of category refers to equality of objects:

Problem

"If $\mathrm{dom}(g)$ is **equal to** $\mathrm{cod}(f)$, then $g \circ f$ exists."

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Problem

"If $\mathrm{dom}(g)$ is **equal to** $\mathrm{cod}(f)$, then $g \circ f$ exists."

Can we give a definition of category without using equality of objects?

. . . and how to fix it.

Solution

Use a logic/language of **dependent sets**, in which $\mathrm{dom}(g) = \mathrm{cod}(f)$ is encoded by what type of thing f and g are.

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A category consists of

- a set O of objects
- for each $x, y \in O$, a type/set $A(x, y)$ of arrows
- for each $x, y, z \in O$ and each $f \in A(x, y)$ and $g \in A(y, z)$, a type/set $g \circ f \in A(x, z)$
- for each $x \in O$, an identity $\mathrm{id}_x \in A(x, x)$
- . . .

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Gives rise to **dependently typed language** by adding logical connectors.

Invariance for statements

Theorem (Freyd '76, Blanc '78)

A property of categories (expressed in 2-sorted first order logic) is invariant under equivalence iff it can be expressed in this dependently typed language, using equality for arrows but not for objects.

Invariance for statements

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- What about **constructions** on categories?

Invariance for statements

Theorem (Freyd '76, Blanc '78)

*A **property** of categories (expressed in 2-sorted first order logic) is invariant under equivalence iff it can be expressed in this dependently typed language, using equality for arrows but not for objects.*

- What about **constructions** on categories?
- What about other mathematical structures?

Outline

What is type theory?

- Type theory is a language for mathematics, akin to category theory.
- Sentences are of the following form:
  - $a_1 : A_1, \ldots, a_n : A_n \vdash B(a_1, \ldots, a_n)$ type
  - $a_1 : A_1, \ldots, a_n : A_n \vdash b(a_1, \ldots, a_n) : B(a_1, \ldots, a_n)$
- e.g.
  - $x, y : \mathsf{ob}\mathscr{C} \vdash \hom_{\mathscr{C}}(x, y)$ type
  - $x : \mathsf{ob}\mathscr{C} \vdash 1_x : \hom_{\mathscr{C}}(x, x)$
- We conflate mathematical objects and mathematical statements.
  - $n : \mathbb{N} \vdash \mathsf{isEven}(n)$ type
  - $n : \mathbb{N} \vdash e(n) : \mathsf{isEven}(2n)$
  - $n : \mathbb{N} \vdash \mathsf{Vect_n}(\mathbb{N})$ type
  - $n : \mathbb{N} \vdash \mathrm{o}(n) : \mathsf{Vect_n}(\mathbb{N})$

Interpretations of type theory

- Examples:
  - $n : \mathbb{N} \vdash \mathsf{isEven}(n)\ \mathsf{type}$
  - $n : \mathbb{N} \vdash e(n) : \mathsf{isEven}(2n)$
  - $n : \mathbb{N} \vdash \mathsf{Vect}_{\mathsf{n}}(\mathbb{N})\ \mathsf{type}$
  - $n : \mathbb{N} \vdash \mathrm{o}(n) : \mathsf{Vect}_{\mathsf{n}}(\mathbb{N})$
- There are many interpretations of dependent type theory:

	Contexts	**Types**	**Terms**
Logical	hypotheses	predicates	proofs
Set theoretic	indices	indexed sets	sections
Homotopical	base space	total space	sections

Type formers

- We can define the natural numbers, booleans, the circle, and coproducts as initial objects in the following way. (Dependent) functions and (dependent) products are defined similarly.

Natural numbers

$$\frac{}{\vdash \mathbb{N} \text{ type}} \qquad \frac{}{\vdash 0 : \mathbb{N}} \qquad \frac{\vdash x : \mathbb{N}}{\vdash sx : \mathbb{N}}$$

$$\frac{x : \mathbb{N} \vdash D(x) \text{ type} \qquad \vdash z : D(0) \qquad x : \mathbb{N}, y : D(x) \vdash \sigma(y) : D(sx)}{\begin{array}{c} x : \mathbb{N} \vdash d(x) : D(x) \\ \vdash d(0) \equiv z : D(0) \qquad x : \mathbb{N} \vdash \sigma(d(x)) \equiv d(sx) : D(sx) \end{array}}$$

Type formers

Binary product

$$\frac{\vdash A \text{ type} \qquad \vdash B \text{ type}}{\vdash A \times B \text{ type}} \qquad \frac{\vdash a : A \qquad \vdash b : B}{\vdash \langle a, b \rangle : A \times B}$$

$$\frac{x : A \times B \vdash D(x) \text{ type} \qquad a : A, b : B \vdash \sigma(a, b) : D\langle a, b \rangle}{\begin{array}{c} x : A \times B \vdash d(x) : D(x) \\ a : A, b : B \vdash \sigma(a, b) \equiv d\langle a, b \rangle : D\langle a, b \rangle \end{array}}$$

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- Set interpretation: $\times$
- Logical interpretation: $\wedge$

Type formers

Dependent sums

$$\frac{a : A \vdash B(a) \text{ type}}{\vdash \Sigma_{a:A} B(a) \text{ type}} \qquad \frac{\vdash a : A \qquad \vdash b : B(a)}{\vdash \langle a, b \rangle : \Sigma_{a:A} B(a)}$$

$$\frac{x : \Sigma_{a:A} B(a) \vdash D(x) \text{ type} \qquad a : A, b : B(a) \vdash \sigma(a,b) : D\langle a, b \rangle}{\begin{array}{c} x : \Sigma_{a:A} B(a) \vdash d(x) : D(x) \\ a : A, b : B(a) \vdash \sigma(a,b) \equiv d\langle a, b \rangle : D\langle a, b \rangle \end{array}}$$

- Set interpretation: $\cup_{a:A} B(a)$
- Logical interpretation: $\exists_{a:A} B(a)$

The surprising type former

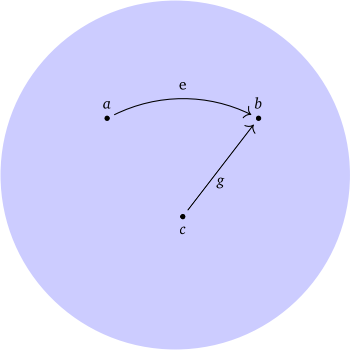
Identity type

$$\frac{\vdash A \text{ type} \qquad \vdash a, b : A}{\vdash a =_A b} \qquad \frac{\vdash A \text{ type} \qquad \vdash a : A}{\vdash \mathrm{refl}_a : a =_A a}$$

$$\frac{\vdash A \text{ type} \qquad x, y : A,\ p : x =_A y \vdash D(p) \text{ type} \qquad x : A \vdash \rho(x) : D(\mathrm{refl}_x)}{\begin{array}{c} x, y : A,\ p : x =_A y \vdash d(p) : D(p) \\ \qquad\qquad x : A \vdash \rho(x) \equiv d(\mathrm{refl}_x) : D(\mathrm{refl}_x) \end{array}}$$

Homotopy type theory

- Equality is given inductively, just like the natural numbers.
- The **equality type** $a = b$ (for two terms $a, b : A$) is generated inductively by the *canonical term* $\mathrm{refl}_a : a = a$ for each term $a : A$.
  - Just as $\mathbb{N}$ is generated by the canonical elements $0 : \mathbb{N}$ and $Sn : \mathbb{N}$ for each $n : \mathbb{N}$.



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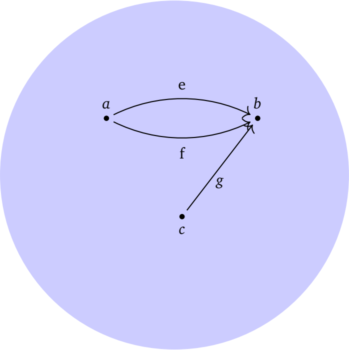

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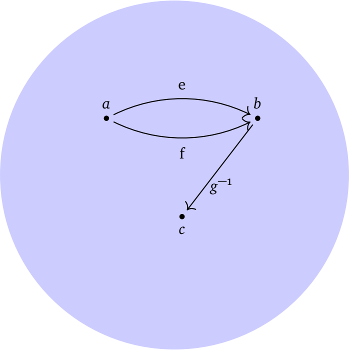
- Equalities are invertible.



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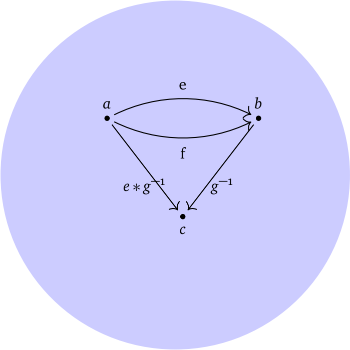
- Equalities are composable.



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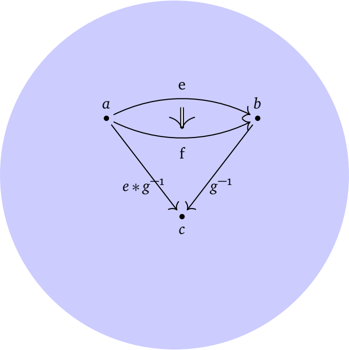
- There can be "higher" equalities.



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- Equalities are invertible.
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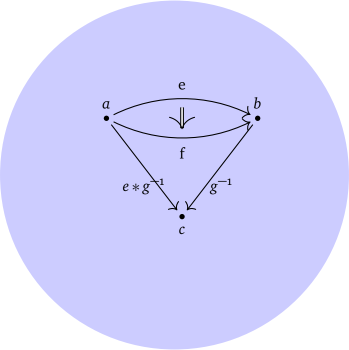
- There can be "higher" equalities.
- This makes types behave like homotopy types or spaces.



Types as Kan complexes

We can interpret

- a type K as a *Kan complex* (space) $[K]$
- a term $k : K$ as a *point* of K
- a dependent type $x : B \vdash E(b)$ as a *Kan fibration* $[p] : [E] \to [B]$
- a dependent term $x : B \vdash e(b) : E(b)$ as a *section* $[e]$ of $[p]$
- a term $p : a =_K b$ as a path from a to b in K

Type formers in Martin-Löf type theory

Type former	Notation	canonical term
Dependent type	$x : A \;\vdash\; B(x)$	
Dependent term	$x : A \;\vdash\; b(x) : B(x)$	
Boolean type	Bool	$\top, \bot$
Natural numbers type	Nat	$0, sx$
Sum type	$\sum_{x:A} B(x)$	(a, b)
Product type	$\prod_{x:A} B(x)$	$\lambda(x : A).b$
Identity type	$x : A, y : A \vdash x = y$	$\mathrm{refl}_x : x = x$
Universe	Type	

Outline

Characterizing equalities

We can characterize the equalities in many type formers.

Theorem about equalities in $\mathbb{N}$

For $n, m : \mathbb{N}$, if $n \equiv m$, then $(n =_{\mathbb{N}} m) \simeq *$; otherwise $(n =_{\mathbb{N}} m) \simeq \emptyset$.

Theorem about equalities in $A \times B$

For $p, q : A \times B$,

$$(p =_{A \times B} q) \simeq (\pi_1 p =_A \pi_1 q) \times (\pi_2 p =_B \pi_2 q).$$

Theorem about equalities in $\Sigma_{a:A} B$

For $p, q : \Sigma_{a:A} B$,

$$(p =_{\Sigma_{a:A} B} q) \simeq \Sigma_{\alpha : \pi_1 p =_A \pi_1 q} \alpha^* \pi_2 p =_{B(\pi_1 q)} \pi_2 q.$$

Under-determined equalities

We could postulate:

Function extensionality

For $f, g : A \to B$, the function

$$(f =_{A \to B} g) \to \Pi_{a:A} f(a) =_B g(a)$$

is an equivalence.

Uniqueness of identity proofs

For $p, q : a =_A b$, we have a term of

$$p =_{a =_A b} q.$$

Univalence

For $A, B : U$, the function

$$(A =_U B) \to (A \simeq_U B)$$

is an equivalence.

Univalent foundations

- *Function extensionality* holds both in the set and the space models (and most other ones).
- *Uniqueness of identity proofs* holds in the set model, but not the space model.
- *Univalence* holds in the space model, but not in the set model.

Univalent foundations admits *univalence* as an axiom (which implies function extensionality).

The equality principle in type theory

Any predicate or construction that can be defined on terms of a type A is of the form $f : A \to B$.

- The predicate "G is an abelian group" is a function $Grp \to Prop$.
- Considering the lattice of subgroups of any group G produces a function $Grp \to Latt$.

Equality principle

We can prove:

$$\prod_{x,y:A} (x = y) \to \prod_{f:A\to B} \big(f(x) = f(y)\big)$$

The equality principle in type theory

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Equality principle

We can prove:

$$\prod_{x,y:A}(x = y) \to \prod_{f:A\to B}\big(f(x) = f(y)\big)$$

Everything respects equality.

Back to the equivalence principle

Using univalence, we have:

Equality to equivalence principle

$$\prod_{x,y:A}(x = y) \to \prod_{B:A\to U}\big(B(x) \simeq B(y)\big)$$

For example:

- The predicate "G is an abelian group" is a function $Grp \to Prop$ which we can compose with the inclusion $Prop \hookrightarrow U$.
- Considering the lattice of subgroups of any group G produces a function $Grp \to Latt$, which we can compose with a forgetful functor $Latt \to U$.

Next time

We would like to prove an equivalence principle like

$$\prod_{G,H:Grp} (G \cong H) \to \prod_{B:Grp \to Latt} B(G) \cong B(H)$$

where $G \cong H$ is group isomorphism and $B(G) \cong B(H)$ is lattice isomorphism.

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To be continued...

Thank you!