Generalizing the equivalence principle in Univalent Foundations to higher categorical structures

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Outline

1. Motivation & review
2. First-order logic with dependent sorts (FOLDS) for lower structures
3. FOLDS categories

Last time

Univalence principle

For a specific type T and suitable notion of equivalence $\simeq$ between terms of T, the canonical map

$$x =_T y \to x \simeq y$$

is an equivalence.

The Univalent Foundations, the univalence axiom is

- an axiom for $T := U$ and $\simeq :=$ equivalence of types (V)
- a theorem for $T := Prop$ and $\simeq :=$ logical equivalence (V)
- a theorem for $T := Set$ and $\simeq :=$ bijection (V)
- a theorem for $T := Grp, Mon, Rng, Proset$ and $\simeq :=$ homomorphisms (CD)
- a theorem for $T := UCat$ and $\simeq :=$ categorical equivalence (AKS)
- a theorem for bicategories, dagger categories, monoidal categories, ...?

Categories

Univalence principle for categories

$$\mathscr{C} =_{UCat} \mathscr{D} \to \mathscr{C} \simeq \mathscr{D}$$

is an equivalence.

Definition

UCat is the type of univalent categories: those categories $\mathscr{C}$ for which

$$(x =_{Ob\mathscr{C}} y) =_U (x \cong y)$$

for every $x, y \in Ob\mathscr{C}$.

Generlize isomorphic → indiscernible

Two objects x, y are isomorphic iff they are 'indiscernible' via category-theoretic operations

Magmas

Magmas

A *magma* is a set M and a binary operation $M \times M \to M$.

There are two notions of 'sameness' for elements m, n of a magma:

(e) Equality: $m =_M n$

(i) Indiscernibility:
$\prod_{x,y:M}(mx = nx) \times (xm = xn) \times \big((xy = m) \leftrightarrow (xy = n)\big)$

This produces two notions of equivalence of magmas:

(e) $M \cong_e N$

(i) $M \cong_i N$

Coquand-Danielsson tells us that $(M =_{Mon} N) \simeq (M \cong_e N)$.
By requiring M, N to be *univalent* (i.e. $e \simeq i$), we then find

$$(M =_{Mon} N) \simeq (M \cong_i N).$$

Goal

Our goal

To define a large class of (higher) *univalent structures* and a notion of *equivalence* between them validating a univalence principle. This then automatically validates a structure identity principle.

Using indiscernibility for the notions of

- univalent
- equivalence

Joint work with Ahrens, Shulman, Tsementzis. arXiv:2102.06275

Outline

First-order logic with dependent sorts (Makkai)

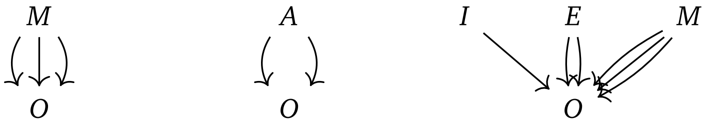
Inverse category

An *inverse category* is a strict category $\mathscr{I}$ and a function $\rho : \mathscr{I} \to \mathsf{Nat}^{\mathsf{op}}$ whose fibers are discrete.

The *height* of an inverse category $(\mathscr{I}, \rho)$ is the maximum value of ρ.

Signatures

Signatures are inverse categories of finite height.



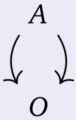
$\mathscr{L}_{\mathsf{Magma}}$ $\mathscr{L}_{\mathsf{Proset}}$ $\mathscr{L}_{\mathsf{Group}}$

Structures

$\mathscr{L}_{\mathsf{Proset}}$-structures

An $\mathscr{L}_{\mathsf{Proset}}$-structure S is

1. A type SO,
2. A type $SA(x,y)$ for every $x,y : O$ (meaning $x \leq y$)



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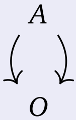
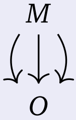


$\mathscr{L}_{\mathsf{Magma}}$-structures

An $\mathscr{L}_{\mathsf{Magma}}$-structure S is

1. A type SO,
2. A type $SM(x,y,z)$ for every $x,y,z : O$ (meaning z is the product of x and y)



We can impose axioms on these structures.

Indiscernibilities

Indiscernibilities between O-elements of $\mathscr{L}_{\mathsf{Proset}}$-structures

An indiscernibility between two terms $p, q : SO$ consists of

- $\prod_{x:SO} SA(p,x) \cong SA(q,x)$
- $\prod_{x:SO} SA(x,p) \cong SA(x,q)$
- $SA(p,p) \cong SA(q,q)$

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Indiscernibilities between O-elements of $\mathscr{L}_{\mathsf{Magma}}$-structures

An indiscernibility between two terms $m, n : SO$ consists of

- $\prod_{x,y:SO} SM(m,x,y) \cong SM(n,x,y)$
- $\prod_{x,y:SO} SM(x,m,y) \cong SM(x,n,y)$
- $\prod_{x,y:SO} SM(x,y,m) \cong SM(x,y,n)$
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Indiscernibilities at the top-level

Indiscernibilities between A-elements of $\mathscr{L}_{\mathsf{Proset}}$-structures

An indiscernibility between two terms $a, b : SA(p, q)$ consists of

- -

so all terms of $a, b : SA(p, q)$ are (trivially) indiscernible.

Definition (univalent structure)

A structure M of a signature $\mathscr{L}$ is *univalent* if the type of indiscernibilities between any two terms of any one sort is equivalent to the type of equalities between them.

In particular, this means that all of the top-level sorts are propositions, and all of the next-level sorts are sets.

Univalent structures

Proposition

A $\mathscr{L}_{\mathsf{Proset}}$-structure S is univalent when each $p \leq q$ is a proposition and $(p = q) \to (p \leq q) \times (q \leq p)$ is an equivalence - in other words, when A is a poset.

Proposition

A $\mathscr{L}_{\mathsf{Magma}}$-structure S is univalent when each $SM(m, n, p)$ is a proposition and $(m = n) \to \prod_{x,y:M}(mx = nx) \times (xm = xn) \times \big((xy = m) \leftrightarrow (xy = n)\big)$ is an equivalence.

Proposition

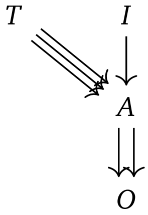
A topological space T is univalent when $(x = y) \to \prod_{U \text{ open in } T}(x \in U \leftrightarrow y \in U)$ is an equivalence – in other words, T is a T_0 space.

Outline

$\mathscr{L}_{\text{cat}}$-structures

We can define the data of a category $\mathscr{C}$ to be

- A type $\mathscr{C}O : \mathscr{U}$
- A family $\mathscr{C}A : \mathscr{C}O \times \mathscr{C}O \to \mathscr{U}$
- A family $\mathscr{C}I : \prod_{(x:\mathscr{C}O)} \mathscr{C}A(x,x) \to \mathscr{U}$
- A family $\mathscr{C}T : \prod_{(x,y,z:\mathscr{C}O)} \mathscr{C}A(x,y) \to \mathscr{C}A(y,z) \to \mathscr{C}A(x,z) \to \mathscr{U}$



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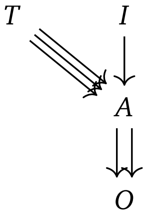

We want to add axioms such as

$$\forall(x,y,z:O).\forall(f:A(x,y)).\forall(g:A(y,z)).\forall(h,h':A(x,z)).$$
$$T(x,y,z,f,g,h) \to T(x,y,z,f,g,h') \to (h = h')$$

(composites are unique), so we add an equality ‘predicate’.

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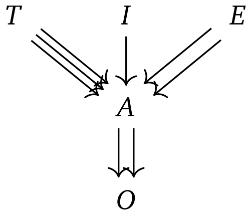

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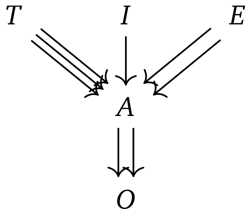

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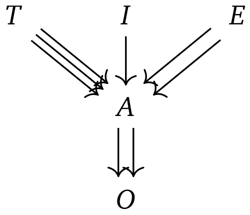

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Univalent $\mathscr{L}_{\mathrm{cat}}$-structures

- Every two elements of $\mathscr{C}I_x(f)$, $\mathscr{C}E_{x,y}(f,g)$, or $\mathscr{C}T_{x,y,z}(f,g,h)$ are indiscernible
  - so each of these types should be a proposition.
- The axioms making E a congruence for T and I make $\mathscr{C}E(f,g)$ the type of indisceribilities between $f,g : \mathscr{C}A(x,y)$
  - so we should have $(f = g) = \mathscr{C}E(f,g)$, making each $\mathscr{C}A(x,y)$ a set.
- The indiscernibilities between $a,b : \mathscr{C}O$ consist of
  1. $\phi_{x\bullet} : \mathscr{C}A(x,a) \simeq \mathscr{C}A(x,b)$ for each $x : \mathscr{C}O$
  2. $\phi_{\bullet z} : \mathscr{C}A(a,z) \simeq \mathscr{C}A(b,z)$ for each $z : \mathscr{C}O$
  3. $\phi_{\bullet\bullet} : \mathscr{C}A(a,a) \simeq \mathscr{C}A(b,b)$
  4. The following for all appropriate w,x,y,z,f,g,h:

$$T_{x,y,a}(f,g,h) \leftrightarrow T_{x,y,b}(f,\phi_{y\bullet}(g),\phi_{x\bullet}(h))$$
$$T_{x,a,z}(f,g,h) \leftrightarrow T_{x,b,z}(\phi_{x\bullet}(f),\phi_{\bullet z}(g),h)$$
$$T_{a,z,w}(f,g,h) \leftrightarrow T_{b,z,w}(\phi_{\bullet z}(f),g,\phi_{\bullet w}(h))$$
$$T_{x,a,a}(f,g,h) \leftrightarrow T_{x,b,b}(\phi_{x\bullet}(f),\phi_{\bullet\bullet}(g),\phi_{x\bullet}(h))$$
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$$T_{a,a,x}(f,g,h) \leftrightarrow T_{b,b,x}(\phi_{\bullet\bullet}(f),\phi_{\bullet x}(g),\phi_{\bullet x}(h))$$
$$T_{a,a,a}(f,g,h) \leftrightarrow T_{b,b,b}(\phi_{\bullet\bullet}(f),\phi_{\bullet\bullet}(g),\phi_{\bullet\bullet}(h))$$

$$I_{a,a}(f) \leftrightarrow I_{b,b}(\phi_{\bullet\bullet}(f))$$
$$E_{x,a}(f,g) \leftrightarrow E_{x,b}(\phi_{x\bullet}(f),\phi_{x\bullet}(g))$$
$$E_{a,x}(f,g) \leftrightarrow E_{b,x}(\phi_{\bullet x}(f),\phi_{\bullet x}(g))$$
$$E_{a,a}(f,g) \leftrightarrow E_{b,b}(\phi_{\bullet\bullet}(f),\phi_{\bullet\bullet}(g))$$

Univalent $\mathscr{L}_{\text{cat}}$-structures continued

Proposition

The type of indiscernibilities between $a, b : \mathscr{C}O$ is equivalent to $a \cong b$.

Proof.

The isomorphisms $\phi_{x\bullet} : \mathscr{C}A(x,a) \cong \mathscr{C}A(x,b)$ are natural by

$$\mathscr{C}T_{x,y,a}(f,g,h) \longleftrightarrow \mathscr{C}T_{x,y,b}(f,\phi_{y\bullet}(g),\phi_{x\bullet}(h))$$

(saying $\phi_{y\bullet}(g) \circ f = \phi_{x\bullet}(g \circ f)$). The rest of the data is redundent.

Thus, in a univalent $\mathscr{L}_{\text{cat}}$-structure, $(a = b) \simeq (a \cong b)$.

Theorem

Univalent $\mathscr{L}_{cat}$-structures are equivalent to the univalent categories of Ahrens-Kapulkin-Shulman.

Categorical equivalences

Theorem (univalence for univalent categories) (Ahrens-Kapulkin-Shulman)

Given univalent categories $\mathscr{C}, \mathscr{D}$,

$$(\mathscr{C} = \mathscr{D}) \simeq (\mathscr{C} \simeq \mathscr{D})$$

A categorial equivalence arises as a very surjective morphism.

A *very surjective morphism* or *equivalence* $F : \mathscr{C} \simeq \mathscr{D}$ of $\mathscr{L}_{\mathrm{cat+E}}$-structures consists of surjections

- $FO : \mathscr{C}O \twoheadrightarrow \mathscr{D}O$
- $FA : \mathscr{C}A(x,y) \twoheadrightarrow \mathscr{D}A(Fx,Fy)$ for every $x,y : \mathscr{C}O$
- $FT : \mathscr{C}T(f,g,h) \twoheadrightarrow \mathscr{D}T(Ff,Fg,Fh)$ for all $f : \mathscr{C}A(x,y), g : \mathscr{C}A(y,z), h : \mathscr{C}A(x,z)$
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- $FT : \mathscr{C}T(f, g, h) \longleftrightarrow \mathscr{D}T(Ff, Fg, Fh)$ for all $f : \mathscr{C}A(x,y), g : \mathscr{C}A(y,z), h : \mathscr{C}A(x,z)$
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Equivalences in general

Definition (equivalence)

An *equivalence* $M \simeq N$ between two $\mathscr{L}$-structures is a very split-surjective morphism $M \to N$.

Theorem

Given two univalent $\mathscr{L}$-structures M and N,

$$(M = N) \simeq (M \simeq N).$$

Theorem

For a signature $L : \mathsf{Sig}(n)$, the type of univalent L-structures is of h-level $n+1$.

Example: magmas

Equivalences of univalent magmas

An equivalence of magmas N, P consists of surjections

- $FO : NO \twoheadrightarrow PO$
- $FM : NM(x, y, z) \twoheadrightarrow PM(Fx, Fy, Fz)$

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Summary

For every signature $\mathscr{L}$, we have

- a notion of indiscernibility within each sort,
- a notion of univalent structures,
- a notion of equivalence,
- a univalence theorem,
- and thus a (higher) structure identity principle.

Summary

For every signature $\mathscr{L}$, we have

- a notion of indiscernibility within each sort,
- a notion of univalent structures,
- a notion of equivalence,
- a univalence theorem,
- and thus a (higher) structure identity principle.

The paper includes examples of

- †-categories,
- presheaves,
- profunctors,
- semi-displayed categories,
- bicategories,
- ...

Thank you!