Homotopical models of type theory

Paige Randall North

The Ohio State University

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Outline

Display map categories

Id-types and weak factorization system

Type-theoretic weak factorization systems

Coherence via universes and the simplicial set model

Outline

Display map categories[1]

Definition

A *display map category* $(\mathcal{C}, \mathcal{D})$ consists of a category $\mathcal{C}$ with a terminal object and a class $\mathcal{D}$ (the *display maps*) of morphisms of $\mathcal{C}$ such that

1. every pullback of every display map exists,
2. every pullback of every display map is a display map, and
3. every map to a terminal object is a display map.

Every display map category is a comprehension category

- The category of contexts is $\mathcal{C}$.
- The category of types $\mathcal{T}$ is the full subcategory of $\mathcal{C}^{\rightarrow}$ spanned by those objects which are in $\mathcal{D}$.
- Comprehension $\mathcal{T} \to \mathcal{C}^{\rightarrow}$ is just the inclusion functor.

[1] Cf. Taylor's *classes of display maps*, Shulman's *type-theoretic fibration categories*, and Joyal's *clans* and *tribes*

Display map categories

- Objects of the category represent contexts.
- Display maps $p : T \to \Gamma$ represent types $\Gamma \vdash T$.
  - We're taking a *fibrational* perspective.
  - Given a point $\gamma : * \to \Gamma$ (which represents a term $\gamma : \Gamma$), the fiber $p^{-1}(\gamma)$ represents the type $T(\gamma)$.

- Conditions 1 and 2: The substitution of a map of contexts $s : \Delta \to \Gamma$ into p is represented by taking the pullback:

$$\begin{array}{ccc} s^* T & \longrightarrow & T \\ \Big\downarrow{\scriptstyle s^* p} & & \Big\downarrow{\scriptstyle p} \\ \Delta & \xrightarrow{s} & \Gamma \end{array}$$

- Condition 3: The terminal object represents the empty context, so contexts are the same as types in the empty context.
- Sections t of p represent terms $\Gamma \vdash t : T$.
  - In the empty context, terms of a type T are just points $* \to T$.

Not a true model of HoTT

- This generally does *not* form a *split* comprehension category.
- But there are strictification theorems that turn any comprehension category into an equivalent split one.
- In this talk, I'm appealing to the strictification theorem of Lumsdaine-Warren (2015) which also turns the following type constructors, which I'll give as only *weakly* stable under substitution/pullback, to their strictly stable counterparts.

Type constructors in display map categories

Consider two display maps in a display map category $(\mathcal{C}, \mathcal{D})$:

$$\begin{array}{ccc} T & & \\ \downarrow{\scriptstyle p} & & \\ \Delta & \xrightarrow{s} & \Gamma \end{array}$$

Σ-types[2]

The Σ-*type* of p and s is $\Sigma_s p := s \circ p$. Then $(\mathcal{C}, \mathcal{D})$ *models* Σ-*types* if $\mathcal{D}$ is closed under composition.

[2] *strong sums* in the sense of Jacobs '90

Type constructors in display map categories

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Π-types[3]

A Π-*type* of p and s is a display map $\Pi_s p$ with codomain Γ and the universal property

$$\mathcal{C}/\Gamma(g, \Pi_s p) \cong \mathcal{C}/\Delta(s^* g, p)$$

for every $g \in \mathcal{C}/\Gamma$. Then $(\mathcal{C}, \mathcal{D})$ *models* Π-*types* if there is a Π-type $\Pi_s p$ for every composable s, p.

[3] *products* in the sense of Jacobs '90

Type constructors in display map categories

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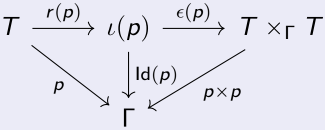
Type constructors in display map categories

Suppose that the ambient display map category $(\mathcal{C}, \mathcal{D})$ models Σ-types. Consider a display map $p : T \to \Gamma$.

Id-types[4]

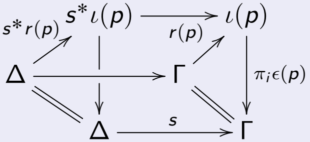
An Id-type of p consists of

- a factorization of the diagonal $p \to p \times p$ in the slice $\mathcal{C}/\Gamma$



such that

- $\epsilon(p)$ is a display map (which makes $\mathrm{Id}(p)$ a display map),
- every pullback of $r(p)$ as shown here has the left lifting property against $\mathcal{D}$.



$(\mathcal{C}, \mathcal{D})$ *models* Id-*types* if there is an Id-type of every display map.

[4] Paulin-Mohring '93, and *weakly stable* in the sense of Lumsdaine-Warren '15

Outline

The Id-type[5]

- The Id-formation rule

$$\frac{\Gamma \vdash T}{\Gamma, t : T, t' : T \vdash \mathsf{Id}_T(t, t')}$$

  requires that there is a display map $\epsilon(p) : \mathsf{Id}(p) \to p \times p$.
- The Id-introduction rule

$$\frac{\Gamma \vdash T}{\Gamma, t : T \vdash r(t) : \mathsf{Id}_T(t, t)}$$

  requires that there is a morphism $r(p) : p \to \mathsf{Id}(p)$.

These two rules require that there is a factorization of the diagonal through a display map. We think of $\mathsf{Id}(p)$ as a path object for p.

$$p \xrightarrow{r(p)} \mathsf{Id}(p) \xrightarrow{\epsilon(p)} p \times p$$

[5] Awodey-Warren '08

The Id-type

- The Martin-Löf Id-elimination rule

$$\frac{\Gamma, t : T, t' : T, q : \mathsf{Id}_T(t, t') \vdash E(t, t', q) \qquad \Gamma, t : T \vdash i(t) : E(t, t, r(t))}{\Gamma, t : T, t' : T, q : \mathsf{Id}_T(t, t') \vdash j(i, t, t', q) : E(t, t', q)}$$

requires that there is a lift in the following diagram in $\mathcal{C}/\Gamma$ making the bottom triangle commute.

$$\begin{array}{ccc} p & \xrightarrow{\quad i \quad} & e \\ {\scriptstyle r(p)}\big\downarrow & & \big\downarrow \\ \mathsf{Id}(p) & = \!= \!= & \mathsf{Id}(p) \end{array}$$

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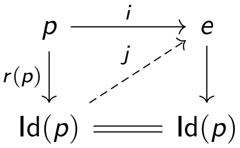

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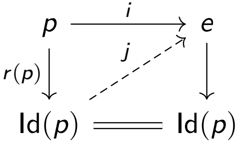

- The Martin-Löf Id-computation rule

$$\frac{\Gamma, t : T, t' : T, q : \mathsf{Id}_T(t,t') \vdash E(t,t',q) \qquad \Gamma \vdash i : E(t,t,r(t))}{\Gamma, t : T \vdash j(i,t,t,r(t)) = i(t) : E(t,t,r(t))}$$

requires that the top triangle commute.

The Id-type

- If we have such lifts, then every $r(p)$ has the left lifting property against every display map.

$$\begin{array}{ccc} p & \xrightarrow{\alpha} & x \\ {\scriptstyle r(p)}\downarrow & & \downarrow{\scriptstyle d} \\ \mathsf{Id}(p) & \xrightarrow[\beta]{} & y \end{array}$$

- The Paulin-Mohring variant asks that this property of $r(p)$ is stable under certain pullbacks.

The Id-type

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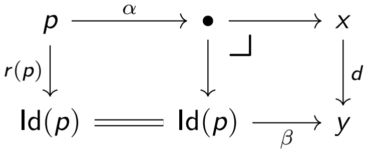

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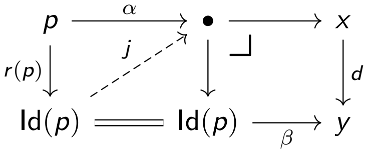


- The Paulin-Mohring variant asks that this property of $r(p)$ is stable under certain pullbacks.

Weak factorization systems

A weak factorization system on a category $\mathcal{C}$ is a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms of $\mathcal{C}$ such that

- every morphism of $\mathcal{C}$ factors into a morphism of $\mathcal{L}$ followed by a morphism of $\mathcal{R}$,
- $\mathcal{L}$ is exactly the class of morphisms with the left lifting property against $\mathcal{R}$, and
- $\mathcal{R}$ is exactly the class of morphisms with the right lifting property against $\mathcal{L}$.

NB: A model category contains two weak factorization systems: $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$.

Weak factorization systems[6]

- The Paulin-Mohring variant is exactly what is needed to generate a weak factorization system.

Consider a category of display maps $(\mathcal{C}, \mathcal{D})$ which models Σ-, Id-types.

- Thinking of the Id-types as path types, we can form the *mapping path space* factorization that takes any morphism $f : X \to Y$ in $\mathcal{C}$ to

$$X \xrightarrow{1\times(r(Y)\circ f)} X \times_Y \mathrm{Id}(Y) \xrightarrow{\pi_Y \circ (f\times 1)^* \epsilon(Y)} Y$$

where $X \times_Y \mathrm{Id}(Y)$ is the pullback

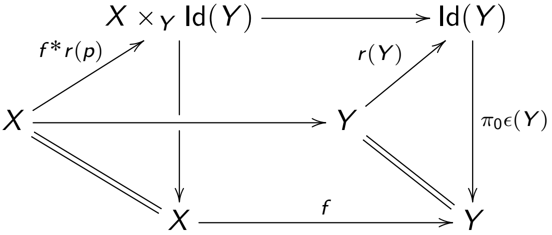
$$\begin{array}{ccc} X \times_Y \mathrm{Id}(Y) & \longrightarrow & \mathrm{Id}(Y) \\ \downarrow & \lrcorner & \downarrow \epsilon \\ X \times Y & \xrightarrow{f\times 1} & Y \times Y \end{array}$$

- The right map is in $\mathcal{D}$ since it is a combination of pullbacks and compositions of display maps.

[6] Gambino-Garner '08

Weak factorization systems[7]

- The left map $X \xrightarrow{1\times(r(Y)\circ f)} X \times_Y \mathrm{Id}(Y)$ has the left lifting property against $\mathcal{D}$ because it can be obtained as the pullback.



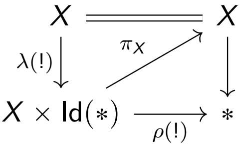
- Our factorization takes a map to one in ${}^{\boxslash}\mathcal{D}$ followed by one of $\mathcal{D}$.
- By formal nonsense, this produces a weak factorization system $({}^{\boxslash}\mathcal{D}, \overline{\mathcal{D}})$ where $\overline{\mathcal{D}}$ is the retract closure of $\mathcal{D}$.

[7] Gambino-Garner '08

Outline

Type-theoretic weak factorization systems[8]

- Every display map category $(\mathcal{C}, \mathcal{D})$ modeling Σ- and Id-types produces a weak factorization system $({}^{\boxslash}\mathcal{D}, \overline{\mathcal{D}})$ on $\mathcal{C}$.
- Which weak factorization systems harbor models of Σ- and Id-types?
- First, if $(\mathcal{C}, \mathcal{D})$ is a display map category modeling Σ- and Id-types, then $(\mathcal{C}, \overline{\mathcal{D}})$ is a display map category modeling Σ- and Id-types (when $\mathcal{C}$ is Cauchy complete).
- Every object in such a weak factorization system must be fibrant (every map to the terminal object must be in the right class).



[8] North '17

Type-theoretic weak factorization systems

Remember: A display map category $(\mathcal{C}, \mathcal{D})$ models Π-types if for every display map p,

1. pullback along p has a partial right adjoint Π_p defined on display maps
2. such that Π_p preserves $\mathcal{D}$.

But 2. is equivalent to

3. pullback along p preserves ${}^{\boxslash}D$.

Definition

A weak factorization system $(\mathcal{L}, \mathcal{R})$ satisfies the *Frobenius condition* if $\mathcal{L}$ is stable under pullback along $\mathcal{R}$.

Say that $(\mathcal{L}, \mathcal{R})$ is *type theoretic* if (1) all objects are fibrant and (2) it satisfies the Frobenius condition.

Type-theoretic weak factorization systems

There is an equivalence between the category of display map categories modeling Σ- and Id-types and the category of type theoretic weak factorization systems on a finitely complete category.

Theorem

Consider a weak factorization system $(\mathcal{L}, \mathcal{R})$ on a finitely complete category $\mathcal{C}$.

$(\mathcal{C}, \mathcal{R})$ is a display map category modeling Σ- and Id-types if and only if (1) every object is fibrant and (2) it satisfies the Frobenius condition.

If for every map r in $\mathcal{R}$, the pullback functor r^* has a partial right adjoint defined on $\mathcal{R}$, then $(\mathcal{C}, \mathcal{R})$ models Π-types.

Given a type theoretic weak factorization system, you recover a model of Id-types by just factoring the diagonal.

Type-theoretic weak factorization systems

Examples from Cisinski model categories:

Suppose you have a model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on a finitely complete category $\mathcal{M}$.

- You can always restrict the wfs $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ to a wfs $(\mathcal{C}_{\mathcal{F}} \cap \mathcal{W}_{\mathcal{F}}, \mathcal{R}_{\mathcal{F}})$ on the full subcategory $\mathcal{M}_{\mathcal{F}}$ of fibrant objects.
- $\mathcal{M}_{\mathcal{F}}$ is closed under pullbacks along morphisms of $\mathcal{R}_{\mathcal{F}}$.
- So $(\mathcal{M}_{\mathcal{F}}, \mathcal{R}_{\mathcal{F}})$ is a display map category modeling Σ-types.
- $\mathcal{W}_{\mathcal{F}}$ is always stable under pullback in $\mathcal{M}_{\mathcal{F}}$.
- In Cisinski model categories[9], $\mathcal{C}$ is the class of monos, and so is always stable under pullback.
- So $(\mathcal{C}_{\mathcal{F}} \cap \mathcal{W}_{\mathcal{F}}, \mathcal{R}_{\mathcal{F}})$ is Frobenius, and $(\mathcal{M}_{\mathcal{F}}, \mathcal{R}_{\mathcal{F}})$ models Id-types.
- Cisinski model categories are locally cartesian closed, so $(\mathcal{M}_{\mathcal{F}}, \mathcal{R}_{\mathcal{F}})$ models Π-types.

[9] Cisinski '06

Type-theoretic weak factorization systems

Cisinski model categories:

- Quillen model structure on sSet (fibrant objects are Kan complexes)
- Joyal model structure on sSet (fibrant objects are quasicategories)
- Cubical sets

Type-theoretic weak factorization systems

Examples from internal reflexive graphs/pseudo-relations:[10]

Suppose you have, for every object X of a category $\mathcal{C}$, a reflexive graph on X:

$$X \xrightarrow{r} \Gamma X \xrightarrow{\epsilon} X \times X$$

It is

- *strictly transitive* when there is a composition $\mu : \Gamma X \times_X \Gamma X \to \Gamma X$ making this into an internal category,
- *strictly connected* when there is connection $\delta : \Gamma X \to \Gamma^2 X$ and a strength $\tau : X \times \Gamma(*) \to \Gamma(X)$ making certain diagrams commute,
- *strictly symmetric* when there is an involution $\iota : \Gamma X \to \Gamma X$ fixing r and switching the endpoints.

[10] van den Berg-Garner '12, North '17

Type-theoretic weak factorization systems

There is an equivalence between the category of transitive, connected, symmetric, reflexive graphs and the category of display map categories modeling Σ- and Id-types.

Theorem

Consider a weak factorization system $(\mathcal{L}, \mathcal{R})$ on a finitely complete category $\mathcal{C}$.

$(\mathcal{C}, \mathcal{R})$ is a display map category modeling Σ- and Id-types if and only if $(\mathcal{L}, \mathcal{R})$ is generated by a transitive, connected, symmetric, reflexive graph.

In this case, the graph is data of the model of Id-types.

Type-theoretic weak factorization systems

Examples of transitive, connected, symmetric, reflexive graphs:

- In groupoids, the underlying graph $\mathcal{G}^{\cong}$ of a groupoid $\mathcal{G}$
- In categories, the underlying graph $\mathcal{C}^{\cong}$ of the core of a category $\mathcal{C}$

Moore path space in topological spaces:[11]

- The naive path space X^I does not have a composition $\mu : X^I \times_X X^I \to X^I$ that is unital (on either side).
- Let $\Gamma(X)$ be the space of paths of any length: $\{(p, r) \in X^{\mathbb{R}_{\geqslant 0}} \times \mathbb{R}_{\geqslant 0} \mid p(s) = p(r) \text{ for all } s \geqslant r\}$
- $r(x) := (c_x, 0)$
- $\epsilon(p, r) := (p(0), p(r))$
- Get the wfs on *Top* whose right maps are the Hurewicz fibrations.

More generally:

- Given a connected, reflexive graph, one can form the free internal groupoid which is then a transitive, connected, symmetric, reflexive graphs.
- Examples: $X^{y(1)}$ in simplicial sets or cubical sets with connections

[11] May '75

Outline

Display map categories

Id-types and weak factorization system

Type-theoretic weak factorization systems

Coherence via universes and the simplicial set model

Groupoids

Our display map category model of Σ-types and Id-types is not strictly stable under substitution.

$$\begin{array}{ccccc} \bullet & \longrightarrow & \bullet & \longrightarrow & T \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ E & \xrightarrow{\tau} & \Delta & \xrightarrow{\sigma} & \Gamma \end{array}$$

There's an easy solution in this case.[12]

- We construct the comprehension category where functors $\Gamma \to Gpd$ represent types dependent on Γ.
- This is a split comprehension category just because composition is associative.

$$E \xrightarrow{\tau} \Delta \xrightarrow{\sigma} \Gamma \xrightarrow{p} T$$

[12] Hofmann-Streicher '98

Universes

Universes

A *system of universes* in a display map category $(\mathcal{C}, \mathcal{D})$ is a collection of display maps $u_i : \tilde{U}_i \to U_i$ such that for every display map $p : T \to \Gamma$, there is a morphism $\alpha : \Gamma \to U_i$ for which p is a pullback $\alpha^* u_i$.

$$\begin{array}{ccc} T & \longrightarrow & \tilde{U}_i \\ {\scriptstyle p}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle u_i} \\ \Gamma & \xrightarrow{\alpha} & U_i \end{array}$$

Local universes model[13]

- There is a left adjoint $(-)_!$ to the inclusion of split comprehension categories into comprehension categories.
- We get a split comprehension category $(\mathcal{C}, \mathcal{D})_!$ from any display map category, $(\mathcal{C}, \mathcal{D})$ whose category of contexts is just $\mathcal{C}$ and whose types in context Δ are pairs of a morphism $\sigma : \Delta \to \Gamma$ and a display map $p : T \to \Gamma$.

$$\begin{array}{ccc} \sigma^* T & \longrightarrow & T \\ {\scriptstyle \sigma^* p}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ \Delta & \xrightarrow{\sigma} & \Gamma \end{array}$$

- It's split because substituting a $\tau : E \to \Delta$ is given by composition.

$$\begin{array}{ccccc} & & & & T \\ & & & & \downarrow{\scriptstyle p} \\ E & \xrightarrow{\tau} & \Delta & \xrightarrow{\sigma} & \Gamma \end{array}$$

[13] Bénabou, Lusmdaine-Warren '15

Simplicial set model[14]

- For every regular cardinal α, there is a universe $u_\alpha : \tilde{U}_\alpha \to U$.
- This classifies Kan fibrations whose fibers each have cardinality $\leqslant \alpha$.
- The universe is *univalent*, meaning that the appropriate notion of sameness between fibers of u_α corresponds to homotopy equivalence between display maps.
- The universe carries the structure of the type formers.

[14] Voevodsky, Kapulkin-Lumsdaine '18

Types in Kan complexes

- Σ is just composition.
- Π is the right adjoint of pullback.
- The Id-type of a Kan complex can be given by $X^{\Delta[1]}$.
- Propositions are either empty Kan complexes or contractible Kan complexes.
- Proposition-truncation is the 0-coskeleton
  - equivalently, add a 1-simplex between any two 0-simplices, a 2-simplex in any triangle of 1-simplices, etc...
- Sets are disjoint unions of contractible Kan complexes.
- Set-truncation is the 1-coskeleton
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- $\mathbb{N}$ is the set $\mathbb{N}$
- The circle S^1 is a fibrant replacement of the simplicial set with one 0-simplex, and one non-degenerate 1-simplex.

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- To do: think about what univalent categories are in simplicial sets.

Thank you!