A higher structure identity principle

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Joint work with Benedikt Ahrens, Michael Shulman, and Dimitris Tsementzis

paigenorth.github.io/lics.pdf

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Outline

Different notions of equality

Synthetic vs. analytic equalities

In Martin-Löf Type Theory, we always have a (*synthetic*) equality type between $a, b : T$

$$a =_T b.$$

Depending on the type T, we might have a type of "*analytic* equalities"

$$a \cong b.$$

A "univalence principle" for this T and this $\cong$ states that

$$(a =_T b) \to (a \cong b)$$

is an equivalence.

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The univalence *axiom* in type theory states that

$$S =_{\mathcal{U}} T \to S \simeq T$$

is an equivalence.

Identicals and indiscernibilites

Identity of indiscernibles

Leibniz: two things are equal when they are *indiscernible* (have the same properties).

$$(a = b) \leftarrow \left(\forall P. P(a) \leftrightarrow P(b)\right)$$

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- This holds in MLTT.

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$$(a =_T b) \leftrightarrow \left(\prod_{P:T\to\mathcal{U}} P(a) \simeq P(b) \right)$$

- This holds in MLTT.
- Given a 'univalence principle' $(a =_T b) \simeq (a \cong b)$, we would find a *structure identity principle* (in the sense of Aczel):

$$(a \cong b) \to \left(\prod_{P:T\to\mathcal{U}} P(a) \simeq P(b) \right).$$

Groups

Theorem: univalence principle for groups (Coquand-Danielsson)

Given two groups G and H,

$$(G =_{\mathsf{Grp}} H) \simeq (G \cong H).$$

Corollary: structure identity principle for groups

$$(G \cong H) \to \prod_{P:\mathsf{Grp}\to\mathcal{U}} (P(G) \simeq P(H)).$$

Categories

Theorem: univalence principle for categories (Ahrens-Kapulkin-Shulman)

Given two *univalent* categories $\mathscr{C}$ and $\mathscr{D}$,

$$(\mathscr{C} =_{\mathsf{Cat}} \mathscr{D}) \simeq (\mathscr{C} \simeq \mathscr{D}).$$

A univalent category is one in which $(x = y) \simeq (x \cong y)$ for all objects x, y.

Corollary: structure identity principle for categories

$$(\mathscr{C} \simeq \mathscr{D}) \to \prod_{P:\mathsf{Cat}\to\mathscr{U}} (P(\mathscr{C}) \simeq P(\mathscr{D})).$$

Result

Our result

To define a large class of (higher) *structures*, a notion of *univalence*, and a notion of *equivalence* between them validating a univalence principle. This then automatically validates a structure identity principle.

Using ideas from:

- *First Order Logic with Dependent Sorts*, Makkai, 1995.
- *Univalent categories and the Rezk completion*, Ahrens, Kapulkin, Shulman, 2015.

Expanded article: arXiv:2004.06572
Expanded talk: youtu.be/aDsY2j1bff4

Examples

- †-categories,
- presheaves,
- profunctors,
- semi-displayed categories,
- bicategories,
- premonoidal categories,
- thunk-force categories,
- double bicategories,
- (symmetric) colored operads,
- duploids,
- ...

Outline

First-order logic with dependent sorts

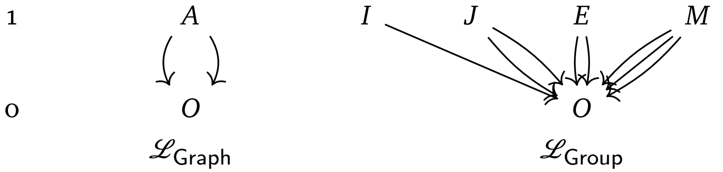
Inverse category

An *inverse category* is a strict category $\mathscr{I}$ and a functor $\rho : \mathscr{I} \to \mathsf{Nat}^{\mathsf{op}}$ whose fibers are discrete.

The *height* of an inverse category $(\mathscr{I}, \rho)$ is the maximum value of ρ.

Signatures

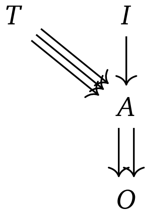
Signatures are inverse categories of finite height.



Example: $\mathscr{L}_{\text{cat}}$-structures

We can define the data of a category $\mathscr{C}$ to be

- A type $\mathscr{C}O : \mathscr{U}$
- A family $\mathscr{C}A : \mathscr{C}O \times \mathscr{C}O \to \mathscr{U}$
- A family $\mathscr{C}I : \prod_{(x:\mathscr{C}O)} \mathscr{C}A(x,x) \to \mathscr{U}$
- A family $\mathscr{C}T : \prod_{(x,y,z:\mathscr{C}O)} \mathscr{C}A(x,y) \to \mathscr{C}A(y,z) \to \mathscr{C}A(x,z) \to \mathscr{U}$



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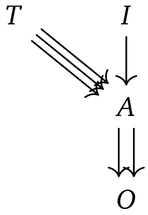

We want to add axioms such as

$$\forall(x,y,z : O).\forall(f : A(x,y)).\forall(g : A(y,z)).\forall(h,h' : A(x,z)).$$
$$T(x,y,z,f,g,h) \to T(x,y,z,f,g,h') \to (h = h')$$

(composites are unique), so we add an equality ‘predicate’.

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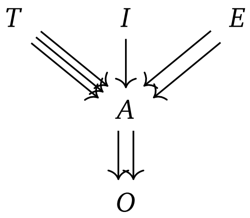

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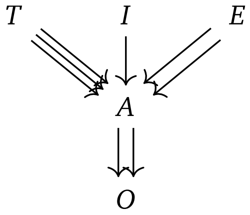

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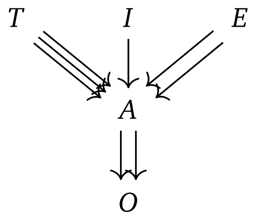

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(composites are unique), so we add an equality 'predicate'.

Univalent FOLDS-categories

Goal

To state the univalence condition

$$(a = b) \simeq (a \cong b)$$

for categories in terms of the the FOLDS structure.

Given $a, b : \mathscr{C}O$, we can define an isomorphism $a \cong b$ using the Yoneda Lemma:

- For each $x : \mathscr{C}O$, an equality $\phi_{x\bullet} : \mathscr{C}A(x, a) = \mathscr{C}A(x, b)$.
- For each $x, y : \mathscr{C}O$, $f : \mathscr{C}A(x, y)$, $g : \mathscr{C}A(y, a)$, and $h : \mathscr{C}A(x, a)$, we have
$$\mathscr{C}T_{x,y,a}(f, g, h) = \mathscr{C}T_{x,y,b}(f, \phi_{y\bullet}(g), \phi_{x\bullet}(h))$$
  $(\phi_{y\bullet}(g) \circ f = \phi_{x\bullet}(g \circ f))$

This is a bit ad hoc and not symmetric.

Indiscernibilites for objects of categories

Instead, can define $a \cong b$ to consist of the following equalities between all the types of our signature with a and b substituted in *all* possible ways:

- For each $x : \mathscr{C}O$, an equality $\phi_{x\bullet} : \mathscr{C}A(x,a) = \mathscr{C}A(x,b)$.
- For each $z : \mathscr{C}O$, an equality $\phi_{\bullet z} : \mathscr{C}A(a,z) = \mathscr{C}A(b,z)$.
- An equality $\phi_{\bullet\bullet} : \mathscr{C}A(a,a) = \mathscr{C}A(b,b)$.
- The following equalities for all appropriate w, x, y, z, f, g, h:

$$T_{x,y,a}(f,g,h) = T_{x,y,b}(f,\phi_{y\bullet}(g),\phi_{x\bullet}(h))$$
$$T_{x,a,z}(f,g,h) = T_{x,b,z}(\phi_{x\bullet}(f),\phi_{\bullet z}(g),h)$$
$$T_{a,z,w}(f,g,h) = T_{b,z,w}(\phi_{\bullet z}(f),g,\phi_{\bullet w}(h))$$
$$T_{x,a,a}(f,g,h) = T_{x,b,b}(\phi_{x\bullet}(f),\phi_{\bullet\bullet}(g),\phi_{x\bullet}(h))$$
$$T_{a,x,a}(f,g,h) = T_{b,x,b}(\phi_{\bullet x}(f),\phi_{x\bullet}(g),\phi_{\bullet\bullet}(h))$$
$$T_{a,a,x}(f,g,h) = T_{b,b,x}(\phi_{\bullet\bullet}(f),\phi_{\bullet x}(g),\phi_{\bullet x}(h))$$
$$T_{a,a,a}(f,g,h) = T_{b,b,b}(\phi_{\bullet\bullet}(f),\phi_{\bullet\bullet}(g),\phi_{\bullet\bullet}(h))$$

$$I_{a,a}(f) = I_{b,b}(\phi_{\bullet\bullet}(f))$$
$$E_{x,a}(f,g) = E_{x,b}(\phi_{x\bullet}(f),\phi_{x\bullet}(g))$$
$$E_{a,x}(f,g) = E_{b,x}(\phi_{\bullet x}(f),\phi_{\bullet x}(g))$$
$$E_{a,a}(f,g) = E_{b,b}(\phi_{\bullet\bullet}(f),\phi_{\bullet\bullet}(g))$$

"Everything above a, b thinks that a and b are the same."

Univalence

We call this an indiscernibility.

Definition (univalent structure)

A structure M of a signature $\mathscr{L}$ is *univalent* if the type of indiscernibilities between any two terms of any one sort is equivalent to the type of equalities between them.

In particular, this means that all of the top-level sorts are propositions, and all of the next-level sorts are sets.

Example: Univalent $\mathscr{L}_{\text{cat}}$-structures

- Every two elements of $\mathscr{C}I_x(f)$, $\mathscr{C}E_{x,y}(f,g)$, or $\mathscr{C}T_{x,y,z}(f,g,h)$ are indiscernible
  - so each of these types should be a proposition.
- The axioms making E a congruence for T and I make $\mathscr{C}E(f,g)$ the type of indisceribilities between $f,g : \mathscr{C}A(x,y)$
  - so we should have $(f = g) = \mathscr{C}E(f,g)$, making each $\mathscr{C}A(x,y)$ a set.
- The type of indiscernibilities between $a,b : \mathscr{C}O$ is $a \cong b$
  - so we should have $(a = b) = (a \cong b)$, making each $\mathscr{C}O$ a groupoid.

Theorem

Univalent $\mathscr{L}_{cat}$-structures are equivalent to the univalent categories of Ahrens-Kapulkin-Shulman.

Categorical equivalences

Theorem (univalence for univalent categories) (AKS 2015)

Given univalent categories $\mathscr{C}, \mathscr{D}$,

$$(\mathscr{C} = \mathscr{D}) \simeq (\mathscr{C} \simeq \mathscr{D})$$

A categorial equivalence are very surjective morphisms.

A *very surjective morphism* or *equivalence* $F : \mathscr{C} \simeq \mathscr{D}$ of $\mathscr{L}_{\text{cat+E}}$-structures consists of surjections

- $FO : \mathscr{C}O \twoheadrightarrow \mathscr{D}O$
- $FA : \mathscr{C}A(x,y) \twoheadrightarrow \mathscr{D}A(Fx, Fy)$ for every $x, y : \mathscr{C}O$
- $FT : \mathscr{C}T(f,g,h) \twoheadrightarrow \mathscr{D}T(Ff, Fg, Fh)$ for all $f : \mathscr{C}A(x,y), g : \mathscr{C}A(y,z), h : \mathscr{C}A(x,z)$
- $FE : \mathscr{C}E(f,g) \twoheadrightarrow \mathscr{D}E(Ff, Fg)$ for all $f, g : \mathscr{C}A(x,y)$
- $FI : \mathscr{C}I(f) \twoheadrightarrow \mathscr{D}I(Ff)$ for all $f : \mathscr{C}A(x,x)$

Outline

Equivalences in general

Definition (equivalence)

An *equivalence* $M \simeq N$ between two $\mathscr{L}$-structures is a very split-surjective morphism $M \to N$.

Theorem

Given two univalent $\mathscr{L}$-structures M and N,

$$(M = N) \simeq (M \simeq N).$$

Theorem

For a signature $L : \mathsf{Sig}(n)$, the type of univalent L-structures is of h-level $n+1$.

Other structures

	Indiscernibilites of objects	Equivalences
Duploids	Neg objects: linear isos; pos objects: thunkable isos	Equivalences up to linear /thunkable isomorphisms
Premonoidal categories	Central isomorphisms	Equivalences up to central isomorphisms
Thunk-force categories	Thunkable isomorphisms	Equivalences up to thunkable isomorphisms
Bicategories	Internal adjoint equivalences	Strong biequivalences
Symmetric colored operads	Isomorphisms	Equivalences
Double bicategories	Invertible companion pairs	...

Summary

For every signature $\mathcal{L}$, we have

- a notion of indiscernibility within each sort,
- a notion of univalent structures,
- a notion of equivalence,
- a univalence theorem,
- and thus a (higher) structure identity principle.

Thank you!