

Towards a type theory for directed homotopy theory

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Outline

Introduction

The hom type former

An interpretation in the category of categories

A homotopical perspective

Conclusion

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Goal

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To develop a directed type theory.

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To formalize theorems about:

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- ▶ Higher category theory

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- ▶ Directed homotopy theory

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- ▶ Directed homotopy theory
 - ▶ Concurrent processes

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 - ▶ Rewriting

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- ▶ Directed paths are introduced as terms of a type former, hom , to be added to Martin-Löf type theory

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- ▶ Directed paths are introduced as terms of a type former, hom , to be added to Martin-Löf type theory
- ▶ Transport along terms of hom

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- ▶ Directed paths are introduced as terms of a type former, hom , to be added to Martin-Löf type theory
- ▶ Transport along terms of hom
- ▶ Independence of hom and Id

What does directed mean?

Syntactically

Martin-Löf's identity type is symmetric/undirected since for any type T , and terms $a, b : T$, there is a function

$$i : \text{Id}_T(a, b) \rightarrow \text{Id}_T(b, a)$$

so that any *path* $p : \text{Id}_T(a, b)$ can be *inverted* to obtain a path $ip : \text{Id}_T(b, a)$.

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- ▶ Can think of these terms as *undirected* paths
- ▶ Can we design a type former of *directed* paths that resembles Id but without its inversion operation i ?

What does directed mean?

Theorem

\mathcal{C} cartesian closed category. A functorial reflexive relation

$$1_{\mathcal{C}} \xrightarrow{r} Id \xrightarrow{\epsilon_0 \times \epsilon_1} 1_{\mathcal{C}} \times 1_{\mathcal{C}}$$

models identity types if and only if it is

1. transitive,
2. homotopical,
3. symmetric.

What does directed mean?

Semantically

higher groupoids

What does directed mean?

Semantically

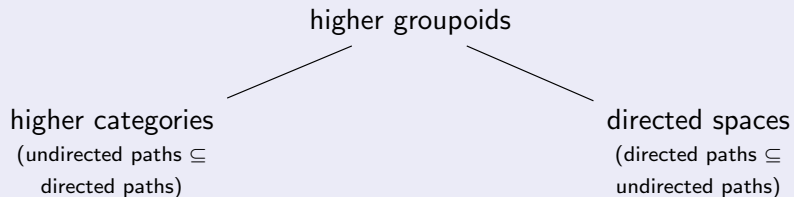
higher categories
(undirected paths \subseteq
directed paths)

higher groupoids



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Rules for hom: core and op

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$$\frac{T \text{ TYPE} \quad t : T^{\text{core}}}{it : T}$$

$$\frac{T \text{ TYPE} \quad t : T^{\text{core}}}{i^{\text{op}}t : T^{\text{op}}}$$

Rules for hom: formation

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Id formation

$$\frac{T \text{ TYPE} \quad s : T \quad t : T}{\text{Id}_T(s, t) \text{ TYPE}}$$

Rules for hom: introduction

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Id introduction

$$\frac{T \text{ TYPE} \quad t : T}{r_t : \text{Id}_T(t, t) \text{ TYPE}}$$

Rules for hom: right elimination and computation

$$\frac{\begin{array}{c} T \text{ TYPE} \quad s : T^{\text{core}}, t : T, f : \text{hom}_T(i^{\text{op}}s, t) \vdash D(f) \text{ TYPE} \\ s : T^{\text{core}} \vdash d(s) : D(1_s) \end{array}}{s : T^{\text{core}}, t : T, f : \text{hom}_T(i^{\text{op}}s, t) \vdash e_R(d, f) : D(f)} \\ s : T^{\text{core}} \vdash e_R(d, 1_s) \equiv d(s) : D(1_s)$$

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Id elimination and computation

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Rules for hom: left elimination and computation

$$\frac{\begin{array}{l} T \text{ TYPE} \quad s : T^{\text{op}}, t : T^{\text{core}}, f : \text{hom}_T(s, it) \vdash D(f) \text{ TYPE} \\ s : T^{\text{core}} \vdash d(s) : D(1_s) \end{array}}{s : T^{\text{op}}, t : T^{\text{core}}, f : \text{hom}_T(s, it) \vdash e_L(d, f) : D(f) \\ s : T^{\text{core}} \vdash e_L(d, 1_s) \equiv d(s) : D(1_s)}$$

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Id elimination and computation

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Syntactic results

- ▶ Transport: for a dependent type $t : T \vdash S(t)$:

$$t : T^{\text{core}}, t' : T, f : \text{hom}_T(i^{\text{op}}t, t'), s : S(it) \\ \vdash \text{transport}_R(s, f) : S(t')$$

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- ▶ Composition: for a type T :

$$\begin{aligned} r : T^{\text{op}}, s : T^{\text{core}}, t : T, f : \text{hom}_T(r, is), g : \text{hom}_T(i^{\text{op}}s, t) \\ \vdash \text{comp}_R(f, g) : \text{hom}_T(r, t) \end{aligned}$$

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The interpretation

- ▶ Use the framework of comprehension categories
- ▶ Dependent types are represented by functors $T : \Gamma \rightarrow \mathit{Cat}$.
- ▶ Dependent terms are represented by natural transformations

$$\begin{array}{ccc} & * & \\ \Gamma & \xrightarrow{\quad} & \mathit{Cat} \\ & \Downarrow t & \\ & T & \end{array}$$

where $* : \Gamma \rightarrow \mathit{Cat}$ is the functor which takes everything to the one-object category.

- ▶ Context extension is represented by the Grothendieck construction which takes each functor $T : \Gamma \rightarrow \mathit{Cat}$ to the Grothendieck opfibration

$$\pi_{\Gamma} : \int_{\Gamma} T \rightarrow \Gamma.$$

Interpreting core and op in the empty context

$$\frac{T \text{ TYPE}}{T^{\text{core}} \text{ TYPE} \quad T^{\text{op}} \text{ TYPE}}$$

$$\frac{T \text{ TYPE} \quad t : T^{\text{core}}}{it : T \quad i^{\text{op}}t : T^{\text{op}}}$$

For any category T ,

- ▶ $T^{\text{core}} := \text{ob}(T)$
- ▶ $T^{\text{op}} := T^{\text{op}}$
- ▶ $i : T^{\text{core}} \rightarrow T$ and $i^{\text{op}} : T^{\text{core}} \rightarrow T^{\text{op}}$ are the identity on objects.

Interpreting hom formation and introduction

$$\frac{T \text{ TYPE} \quad s : T^{\text{op}} \quad t : T}{\text{hom}_T(s, t) \text{ TYPE}}$$

$$\frac{T \text{ TYPE} \quad t : T^{\text{core}}}{1_t : \text{hom}_T(i^{\text{op}}t, it) \text{ TYPE}}$$

For any category T ,

- ▶ Take the functor

$$\text{hom} : T^{\text{op}} \times T \rightarrow \text{Set} \hookrightarrow \text{Cat}.$$

- ▶ Take the natural transformation

$$\begin{array}{ccc} T^{\text{core}} & \begin{array}{c} \xrightarrow{\quad * \quad} \\ \Downarrow \mathbf{1} \\ \xrightarrow{\quad \text{hom} \circ (i^{\text{op}} \times i) \quad} \end{array} & \text{Cat} \end{array}$$

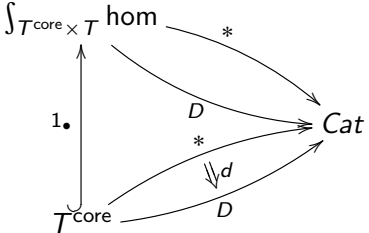
where each component $1_t : * \rightarrow \text{hom}(t, t)$ picks out the identity morphism of t .

Interpreting right hom elimination and computation

$$\frac{\begin{array}{l} T \text{ TYPE} \quad s : T^{\text{core}}, t : T, f : \text{hom}_T(i^{\text{op}}s, t) \vdash D(f) \text{ TYPE} \\ s : T^{\text{core}} \vdash d(s) : D(1_s) \end{array}}{s : T^{\text{core}}, t : T, f : \text{hom}_T(i^{\text{op}}s, t) \vdash e_R(d, f) : D(f)} \\ s : T^{\text{core}} \vdash e_R(d, 1_s) \equiv d(s) : D(1_s)$$

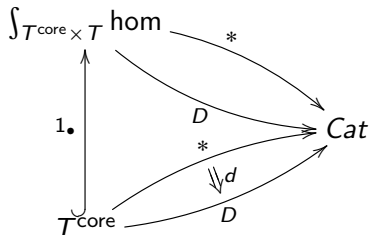
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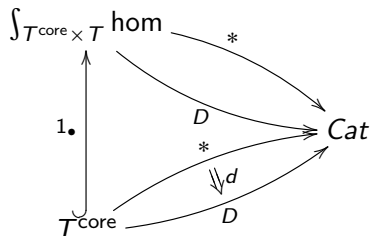
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- Use the fact that the subcategory T^{core} is coreflective:

Interpreting right hom elimination and computation

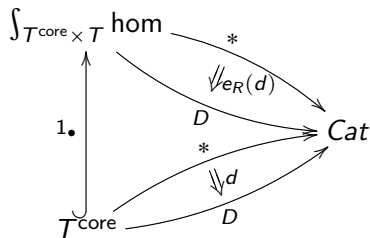
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- ▶ Use the fact that the subcategory T^{core} is coreflective:
 - ▶ for every $(s, t, f) \in \int_{T^{\text{core}} \times T} \text{hom}$ there is a unique morphism $(1_s, f) : (s, s, 1_s) \rightarrow (s, t, f)$ with domain in T^{core}

Interpreting right hom elimination and computation

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- ▶ Set $e_R(d)_{(s,t,f)} := D(1_s, f)d_{(s,s,1_s)}$

Interpreting left hom elimination and computation

$$\frac{\begin{array}{c} T \text{ TYPE} \quad s : T^{\text{op}}, t : T^{\text{core}}, f : \text{hom}_T(s, it) \vdash D(f) \text{ TYPE} \\ s : T^{\text{core}} \vdash d(s) : D(1_s) \end{array}}{s : T^{\text{op}}, t : T^{\text{core}}, f : \text{hom}_T(s, it) \vdash e_L(d, f) : D(f)} \\ s : T^{\text{core}} \vdash e_L(d, 1_s) \equiv d(s) : D(1_s)$$

- ▶ Replace T by T^{op} and apply right hom elimination and computation.

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A homotopical perspective

While the homotopy theory of isomorphisms in categories

$$\mathcal{C} \rightarrow \mathcal{C}^{(\cong)} \rightarrow \mathcal{C} \times \mathcal{C}$$

provides an interpretation of Martin-Löf's identity type, the homotopy theory of morphisms in categories

$$\mathcal{C} \rightarrow \mathcal{C}^{(\rightarrow)} \rightarrow \mathcal{C} \times \mathcal{C}$$

provides an interpretation of this hom former.

The weak factorization system

- ▶ Let (\cong) denote the category with two objects and one isomorphism between them.
- ▶ Let (\rightarrow) denote the category with two objects and one morphism between them.
- ▶ Then factorize the codiagonal of the one-point category in two ways

$$* + * \rightarrow (\cong) \rightarrow * \qquad * + * \rightarrow (\rightarrow) \rightarrow *$$

- ▶ which produces a factorization of any diagonal in two ways which each generate weak factorization systems.

$$\mathcal{C} \rightarrow \mathcal{C}^{(\cong)} \rightarrow \mathcal{C} \times \mathcal{C} \qquad \mathcal{C} \rightarrow \mathcal{C}^{(\rightarrow)} \rightarrow \mathcal{C} \times \mathcal{C}$$

- ▶ The first gives an interpretation of the *Id* type in *Cat*.
- ▶ The second underlies this interpretation of the *hom* type in *Cat*.

The weak factorization system continued

- ▶ The right class of this weak factorization system are those functors $p : E \rightarrow B$ which have the enriched right lifting property

$$\begin{array}{ccc}
 * & \longrightarrow & E \\
 \text{DOM} \downarrow & \nearrow & \downarrow p \\
 (\rightarrow) & \longrightarrow & B
 \end{array}$$

- ▶ so all Grothendieck opfibrations (dependent projections) are in the right class.
- ▶ The functor $1_{\bullet} : T^{\text{core}} \hookrightarrow \int_{T^{\text{core}} \times T} \text{hom}$ is the left part of the factorization of

$$i : T^{\text{core}} \rightarrow T.$$

- ▶ Then the right hom elimination and computation rule arises from the weak factorization system.

$$\begin{array}{ccc}
 T^{\text{core}} & \xrightarrow{d} & \int_{T^{\text{core}} \times T} \text{hom}^D \\
 \downarrow 1_{\bullet} & \nearrow e_R(d) & \downarrow \pi \\
 \int_{T^{\text{core}} \times T} \text{hom} & \xlongequal{\quad} & \int_{T^{\text{core}} \times T} \text{hom}
 \end{array}$$

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Summary & future work

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Future work

We need to:

- ▶ integrate this into traditional Martin-Löf type theory
 - ▶ integrate Id and hom in the same theory
 - ▶ specify Σ , Π , etc

Summary & future work

Summary

We have:

- ▶ a directed type theory
- ▶ with a model in Cat .

Future work

We need to:

- ▶ integrate this into traditional Martin-Löf type theory
 - ▶ integrate Id and hom in the same theory
 - ▶ specify Σ , Π , etc
- ▶ find interpretations in categories of directed spaces
 - ▶ build 'directed' weak factorization systems
 - ▶ build universes

Thank you!

Further Reading



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