Directed homotopy type theory

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Outline

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To develop a directed type theory.

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To formalize theorems about:

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    - Concurrent processes
    - Rewriting
    - Neural networks
    - ...

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The Id-type

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In MLTT, the *identity type* internalizes the notion of equality.

- There is a type $\mathrm{Id}_T(a, b)$ for any type T and $a, b : T$
- Inductively generated by $\mathrm{refl}_a : \mathrm{Id}_T(a, a)$

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We can show:

- The relation $\mathrm{Id}_T(a, b)$ is (reflexive), transitive, and symmetric
- The identity type can be iterated:
  $p, q : \mathrm{Id}_T(a, b), \quad \alpha, \beta : \mathrm{Id}_{\mathrm{Id}_T(a,b)}(p, q), \quad \ldots$
- It is possible for $\mathrm{Id}_{\mathrm{Id}_T(a,b)}(p, q)$ to be empty.

Thus the Id-type constructor endows each type with the structure of an ∞-groupoid, or *space*.

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Thus the Id-type constructor endows each type with the structure of an ∞-groupoid, or *space*.

$\rightarrow$ homotopy type theory

What does directed mean? (Syntactically)

Id-type is symmetric/undirected

For any type T, and terms $a, b : T$, there is a function

$$i : \mathrm{Id}_T(a, b) \to \mathrm{Id}_T(b, a)$$

so that any *path* $p : \mathrm{Id}_T(a, b)$ can be *inverted* to obtain a path $ip : \mathrm{Id}_T(b, a)$.

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- Can think of identity terms as *undirected* paths
- Can we design a type former of *directed* paths that resembles Id but without its inversion operation i?

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∞-groupoids

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Directed spaces

Rough definition

A space together with a subset of its paths that are marked as 'directed'

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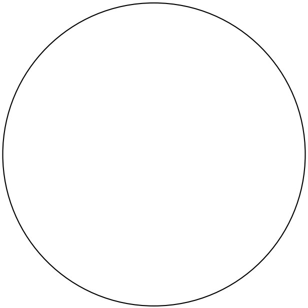

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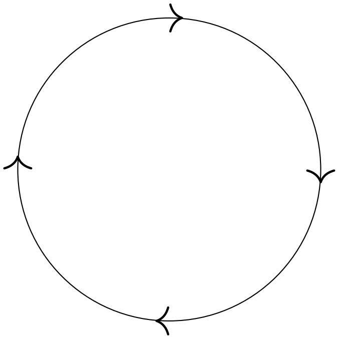

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Concurrent processes can be represented by directed spaces.

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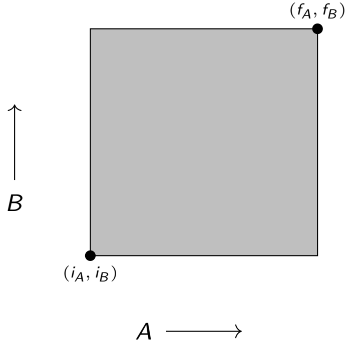

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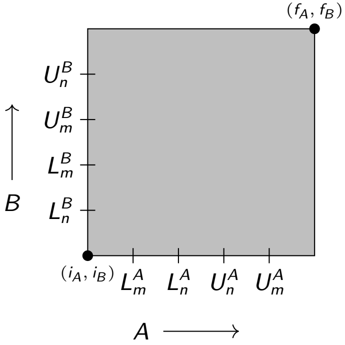

- A, B are two processes
- m, n are two memory locations
- which can be locked (L) or unlocked (U) by each process

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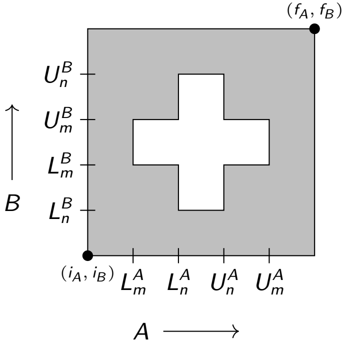

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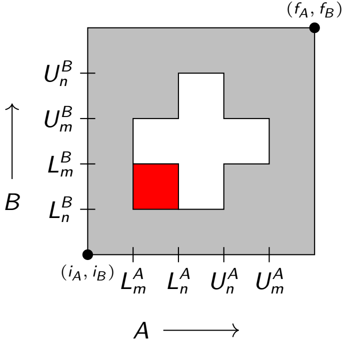

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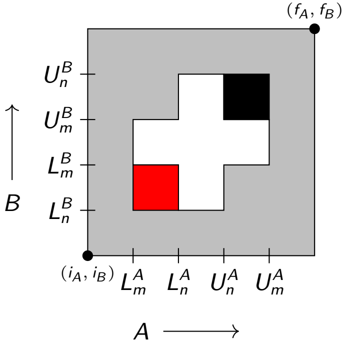

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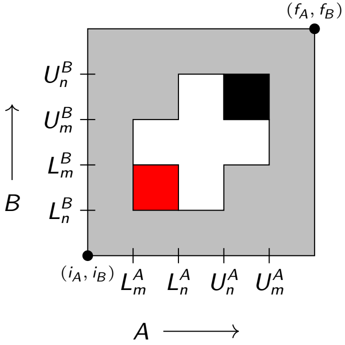

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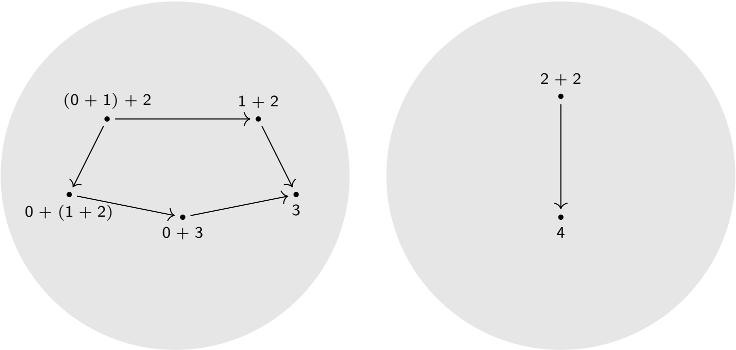
- which can be locked (L) or unlocked (U) by each process

Fundamental questions:

- Which states are safe?
- Which states are reachable?

Application: Term rewriting systems

Consider expressions in the monoid $N = (\mathbb{N}, 0, +)$.



- Interested in families $D(n)$ indexed by $n \in N$ for which rewrite rules $n \to m$ induce rewrites $D(n) \to D(m)$

Outline

Introduction

Directed homotopy theory

A first attempt (the hom-type former)

A second attempt (modal version)

The interpretations

	Id	hom
Types	Groupoids	Categories
Dependent types	$T : \Gamma \to Grp$	$T : \Gamma \to Cat$
Dependent terms	$1 \Rightarrow T : \Gamma \to Grp$	$1 \Rightarrow T : \Gamma \to Grp$
Context extension	$\int_{\gamma \in \Gamma} T(\gamma)$	$\int_{\gamma \in \Gamma} T(\gamma)$
T^{op}	-	T^{op}
T^{core}	-	T^{core}
Id / hom	$\mathrm{hom} : T^{\mathrm{op}} \times T \to \mathrm{Set}$	$\mathrm{hom} : T \times T \to \mathrm{Set}$
$\mathrm{refl}_a : \mathrm{Id}(a, a)$ / $1_a : \mathrm{hom}(a, a)$	identity	identity

- **Universal property of** Id: $\int_{(x,y) \in T \times T} \mathrm{hom}(x, y)$ is freely generated by the identities
- **Universal property of** hom: $\int_{(x,y) \in T^{\mathrm{op}} \times T} \mathrm{hom}(x, y)$ is 'doubly' freely generated by the identities. There are two arrows from an identity to an f: postcomposing 1_x with f and precomposing 1_y with f.

Rules for hom: core and op

$$\frac{T \text{ type}}{T^{\text{core}} \text{ type}}$$

$$\frac{T \text{ type}}{T^{\text{op}} \text{ type}}$$

$$\frac{T \text{ type} \qquad t : T^{\text{core}}}{it : T}$$

$$\frac{T \text{ type} \qquad t : T^{\text{core}}}{i^{\text{op}}t : T^{\text{op}}}$$

Rules for hom: formation

Id formation

$$\frac{T \ \ \text{type} \qquad s : T \qquad t : T}{\mathsf{Id}_T(s,t) \ \ \text{type}}$$

hom formation

$$\frac{T \ \ \text{type} \qquad s : T^{\mathrm{op}} \qquad t : T}{\hom_T(s,t) \ \ \text{type}}$$

Rules for hom: introduction

Id introduction

$$\frac{T \quad \text{type} \qquad t : T}{r_t : \mathrm{Id}_T(t, t) \quad \text{type}}$$

hom introduction

$$\frac{T \quad \text{type} \qquad t : T^{\mathrm{core}}}{1_t : \hom_T(i^{\mathrm{op}} t, it) \quad \text{type}}$$

Rules for hom: right elimination and computation

Id elimination and computation

$$\frac{s : T, t : T, f : \mathrm{Id}_T(s,t) \vdash D(f) \ \text{ type} \qquad s : T \vdash d(s) : D(r_s)}{\begin{array}{c} s : T, t : T, f : \mathrm{Id}_T(s,t) \vdash j(d,f) : D(f) \\ s : T \vdash j(d, r_s) \equiv d(s) : D(r_s) \end{array}}$$

hom right elimination and computation

$$\frac{\begin{array}{c} s : T^{\mathrm{core}}, t : T, f : \hom_T(i^{\mathrm{op}}s, t) \vdash D(f) \ \text{ type} \\ s : T^{\mathrm{core}} \vdash d(s) : D(1_s) \end{array}}{\begin{array}{c} s : T^{\mathrm{core}}, t : T, f : \hom_T(i^{\mathrm{op}}s, t) \vdash e_R(d,f) : D(f) \\ s : T^{\mathrm{core}} \vdash e_R(d, 1_s) \equiv d(s) : D(1_s) \end{array}}$$

Rules for hom: left elimination and computation

Id elimination and computation

$$\frac{s : T, t : T, f : \mathrm{Id}_T(s,t) \vdash D(f) \ \text{type} \qquad s : T \vdash d(s) : D(r_s)}{\begin{array}{c} s : T, t : T, f : \mathrm{Id}_T(s,t) \vdash j(d,f) : D(f) \\ s : T \vdash j(d, r_s) \equiv d(s) : D(r_s) \end{array}}$$

hom left elimination and computation

$$\frac{\begin{array}{c} s : T^{\mathrm{op}}, t : T^{\mathrm{core}}, f : \hom_T(s, it) \vdash D(f) \ \text{type} \\ s : T^{\mathrm{core}} \vdash d(s) : D(1_s) \end{array}}{\begin{array}{c} s : T^{\mathrm{op}}, t : T^{\mathrm{core}}, f : \hom_T(s, it) \vdash e_L(d,f) : D(f) \\ s : T^{\mathrm{core}} \vdash e_L(d, 1_s) \equiv d(s) : D(1_s) \end{array}}$$

Syntactic results

- Transport: for a dependent type $t : T \vdash S(t)$:

$$t : T^{\mathrm{core}}, t' : T, f : \hom_T(i^{\mathrm{op}} t, t'), s : S(it) \\ \vdash \mathrm{transport}_{\mathrm{R}}(s, f) : S(t')$$

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- Composition: for a type T:

$$r : T^{\mathrm{op}}, s : T^{\mathrm{core}}, t : T, f : \hom_T(r, is), g : \hom_T(i^{\mathrm{op}} s, t) \\ \vdash \mathrm{comp}_{\mathrm{R}}(f, g) : \hom_T(r, t)$$

Outline

Problems with the first attempt

The functions $\mathrm{op}, \mathrm{core}$ are problematic.

- There are no introduction rules for T^{core} or T^{op}
- Including the identity type causes the hom type to collapse to the identity type on elements of T^{core}.
- We need a op function on the universe; e.g. the **1-functor** $\mathrm{op} : \mathit{Cat} \to \mathit{Cat}$. This does not exist for 2-categories and up.

Modal directed homotopy type theory

Solution

The solution is to properly account for core, op, etc.

- **syntactically**: modal type theory
- **semantically**: multisided weak factorization systems (jww van den Berg, McCloskey)

(The theory of multisided weak factorization systems accounts for multiple fibrations – e.g. Grothendieck fibrations, opfibrations, isofibrations – in one category and how they interact, inspired by the two-sided fibrations of Street.)

Modal directed type theory

The idea:

- Forget about having type constructors $T \mapsto T^{\mathrm{op}}, T^{\mathrm{core}}$

$$x : R^{\mathrm{op}}, y : S^{\mathrm{core}} \vdash T$$

- Instead op and core should be descriptions of how variables can be used.

$$x \overset{-}{:} R, y \overset{\circ}{:} S \vdash T$$

A modal approach

Following Licata-Shulman-Riley's (2017) modal framework for the sequent calculus.

- four modes that form a lattice:

$$\begin{tikzcd} & \times \arrow[dl, no head] \arrow[dr, no head] & \\ - \arrow[dr, no head] & & + \arrow[dl, no head] \\ & \circ & \end{tikzcd}$$

- with an multiplication
  - $+$ is the identity
  - $\circ$ is almost an absorbing element: $\circ a = a \circ = \circ$ except $\circ \times = \times$
  - $\times$ is almost an absorbing element: $\times a = a \times = \times$ except $\times \circ = \circ$
  - $-$ is a root of unity: $-- = +$.
- This is a sub-monoidal category of $[\mathrm{Cat}, \mathrm{Cat}]$

Orientations

Contexts are annotated by *orientations*. We write:

$$w \overset{\times}{:} A, x \overset{+}{:} B, y \overset{-}{:} C, z \overset{\circ}{:} D \vdash T$$

or

$$w : A, x : B, y : C, z : D \vdash_{x, y^{-}, z^{\circ}} T$$

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Orientations on contexts inherit an order from the lattice, so we use the following rule.

$$\frac{\Gamma \vdash_\ell \mathcal{J} \qquad m \leqslant \ell \ \ \Gamma\text{-ort}}{\Gamma \vdash_m \mathcal{J}} \ \textsc{Ort-Subst}$$

Structural rules

$$\frac{\Gamma, x : \sigma, \Delta \ \mathsf{ctx}}{\Gamma, x : \sigma, \Delta \vdash_x x : \sigma} \ \textsc{Var}$$

$$\frac{\Gamma, \Delta \vdash_{\ell,m} \mathcal{J} \qquad \Gamma \vdash_\ell \rho \ \mathsf{type}}{\Gamma, x : \rho, \Delta \vdash_{\ell,m} \mathcal{J}} \ \textsc{Weak}$$

$$\frac{\begin{array}{c}\Gamma, x : \rho, \Delta \vdash_{\ell,\omega,m} T \\ \Gamma \vdash_\ell U : \rho \qquad n \leqslant \ell \ \ \Gamma\text{-ort} \qquad n \leqslant \omega \cdot \ell \ \ \Gamma\text{-ort}\end{array}}{\Gamma, \Delta[U/x] \vdash_{n,m} T[U/x]} \ \textsc{Subst}$$

The new hom-type

hom formation

$$\frac{\Gamma \vdash_{\ell} A \text{ type}}{\Gamma, x : A, y : A \vdash_{\ell, x^{-}, y} \mathsf{hom}_A(x, y) \text{ type}} \text{ hom-}\textsc{form}$$

hom introduction

$$\frac{\Gamma \vdash_{\ell} A \text{ type}}{\Gamma, x : A \vdash_{\ell, x^{\circ}} 1_x : \mathsf{hom}_A(x, x) \text{ type}} \text{ hom-}\textsc{intro}$$

The new Id-type

Id° formation

$$\frac{\Gamma \vdash_{\ell} A \text{ type}}{\Gamma, x : A, y : A \vdash_{\ell, x^{\circ}, y^{\circ}} \mathsf{Id}^{\circ}_{A}(x, y) \text{ type}} \ \mathsf{Id}^{\circ}\text{-}\textsc{form}$$

Id° introduction

$$\frac{\Gamma \vdash_{\ell} A \text{ type}}{\Gamma, x : A \vdash_{\ell, x^{\circ}} \mathsf{refl}_{x} : \mathsf{Id}^{\circ}_{A}(x, x) \text{ type}} \ \mathsf{Id}^{\circ}\text{-}\textsc{intro}$$

The new Id-type

$\mathrm{Id}^\times$ formation

$$\frac{\Gamma \vdash_\ell A \text{ type}}{\Gamma, x : A, y : A \vdash_{\ell, x^\times, y^\times} \mathrm{Id}^\times(x, y) \text{ type}} \ \mathrm{Id}^\times\text{-}\textsc{form}$$

$\mathrm{Id}^\times$ introduction

$$\frac{\Gamma \vdash_\ell A \text{ type}}{\Gamma, x : A \vdash_{\ell, x^\times} \mathrm{refl}_x : \mathrm{Id}^\times(x, x) \text{ type}} \ \mathrm{Id}^\times\text{-}\textsc{intro}$$

Inside the type theory

What can we do?

- Find inclusions $\mathrm{Id}^{\circ}(a, b) \to \hom(a, b) \to \mathrm{Id}^{\times}(a, b)$, but not $\hom(a, b) \to \mathrm{Id}^{\circ}(a, b)$.
- Transport and compose.

What can't we do?

- Form all Σ types (F types in LSR). For example, the one you should get from $a : A \vdash_{a \vdash \mathrm{op}(a)} 1$ is A^{op}.

Future work

- Connect this formally with the intended semantics (jww van den Berg-McCloskey and Ahrens-van der Weide)
- Understand which Σ types exist.
- Π-types, directed univalence, higher inductive types, etc...

Thank you!