Approaches to directed homotopy type theory

Paige Randall North

Utrecht University

5 July 2023

Outline

Goal

Goal

To develop a directed type theory.

Goal

Goal

To develop a directed type theory.

To formalize theorems about:

Goal

Goal

To develop a directed type theory.

To formalize theorems about:

- Higher category theory

Goal

Goal

To develop a directed type theory.

To formalize theorems about:

- Higher category theory
- Directed homotopy theory

Goal

Goal

To develop a directed type theory.

To formalize theorems about:

- Higher category theory
- Directed homotopy theory
  - Concurrent processes (plugging into Fajstrup-Goubault-Haucourt-Mimram-Raussen)
  - Rewriting
  - Neural networks
  - ...

Goal

Goal

To develop a directed type theory.

To formalize theorems about:

- Higher category theory
- Directed homotopy theory
  - Concurrent processes (plugging into Fajstrup-Goubault-Haucourt-Mimram-Raussen)
  - Rewriting
  - Neural networks
  - ...

The Id-type

Id-type

In MLTT, the *identity type* internalizes the notion of equality.

- There is a type $\mathsf{Id}_T(a, b)$ for any type T and $a, b : T$
- Inductively generated by $\mathsf{refl}_a : \mathsf{Id}_T(a, a)$

The Id-type

Id-type

In MLTT, the *identity type* internalizes the notion of equality.

- There is a type $\mathsf{Id}_T(a, b)$ for any type T and $a, b : T$
- Inductively generated by $\mathsf{refl}_a : \mathsf{Id}_T(a, a)$

We can show:

- The relation $\mathsf{Id}_T(a, b)$ is (reflexive), transitive, and symmetric
- The identity type can be iterated:
  $p, q : \mathsf{Id}_T(a, b), \quad \alpha, \beta : \mathsf{Id}_{\mathsf{Id}_T(a,b)}(p, q), \quad \ldots$
- It is possible for $\mathsf{Id}_{\mathsf{Id}_T(a,b)}(p, q)$ to be empty.

Thus each type gets the structure of an ∞-groupoid, or *space*.

The Id-type

Id-type

In MLTT, the *identity type* internalizes the notion of equality.

- There is a type $\mathsf{Id}_T(a,b)$ for any type T and $a,b:T$
- Inductively generated by $\mathsf{refl}_a : \mathsf{Id}_T(a,a)$

We can show:

- The relation $\mathsf{Id}_T(a,b)$ is (reflexive), transitive, and symmetric
- The identity type can be iterated:
  $p,q : \mathsf{Id}_T(a,b), \quad \alpha,\beta : \mathsf{Id}_{\mathsf{Id}_T(a,b)}(p,q), \quad \ldots$
- It is possible for $\mathsf{Id}_{\mathsf{Id}_T(a,b)}(p,q)$ to be empty.

Thus each type gets the structure of an ∞-groupoid, or *space*.

Can we find a type former that resembles Id, i.e. is reflexive and transitive, but *not symmetric*?

Directed spaces

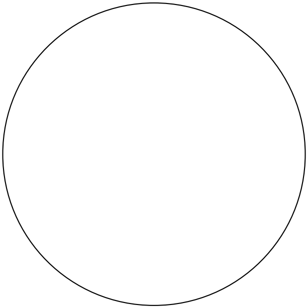
Rough definition

A space together with a subset of its paths that are marked as 'directed'

Directed spaces

Rough definition

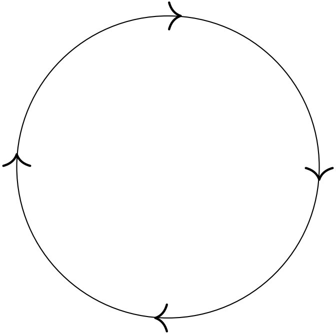
A space together with a subset of its paths that are marked as ‘directed’



Directed spaces

Rough definition

A space together with a subset of its paths that are marked as 'directed'



Rules for hom: core and op

$$\frac{T \ \text{type}}{T^{\text{core}} \ \text{type}}$$

$$\frac{T \ \text{type}}{T^{\text{op}} \ \text{type}}$$

$$\frac{T \ \text{type} \qquad t : T^{\text{core}}}{it : T}$$

$$\frac{T \ \text{type} \qquad t : T^{\text{core}}}{i^{\text{op}} t : T^{\text{op}}}$$

The interpretations

	Id	hom
Types	Groupoids	Categories
Dependent types	$T : \Gamma \to Grp$	$T : \Gamma \to Cat$
Dependent terms	$1 \Rightarrow T : \Gamma \to Grp$	$1 \Rightarrow T : \Gamma \to Grp$
Context extension	$\int_{\gamma \in \Gamma} T(\gamma)$	$\int_{\gamma \in \Gamma} T(\gamma)$
T^{op}	-	T^{op}
T^{core}	-	T^{core}
Id / hom	$\mathrm{hom} : T^{\mathrm{op}} \times T \to \mathrm{Set}$	$\mathrm{hom} : T \times T \to \mathrm{Set}$
$\mathrm{refl}_a : \mathrm{Id}(a, a)$ / $1_a : \mathrm{hom}(a, a)$	identity	identity

- **Universal property of** Id: $\int_{(x,y) \in T \times T} \mathrm{hom}(x, y)$ is freely generated by the identities
- **Universal property of** hom: $\int_{(x,y) \in T^{\mathrm{op}} \times T} \mathrm{hom}(x, y)$ is 'double' freely generated by the identities. There are two arrows from an identity to an f: postcomposing 1_x with f and precomposing 1_y with f.

Rules for hom: formation

Id formation

$$\frac{T \ \ \text{type} \qquad s : T \qquad t : T}{\mathsf{Id}_T(s, t) \ \ \text{type}}$$

hom formation

$$\frac{T \ \ \text{type} \qquad s : T^{\mathsf{op}} \qquad t : T}{\mathsf{hom}_T(s, t) \ \ \text{type}}$$

Rules for hom: introduction

Id introduction

$$\frac{T \ \ \text{type} \qquad t : T}{r_t : \mathrm{Id}_T(t, t) \ \ \text{type}}$$

hom introduction

$$\frac{T \ \ \text{type} \qquad t : T^{\text{core}}}{1_t : \hom_T(i^{\text{op}} t, it) \ \ \text{type}}$$

Rules for hom: right elimination and computation

Id elimination and computation

$$\frac{s : T, t : T, f : \mathrm{Id}_T(s,t) \vdash D(f) \ \text{type} \qquad s : T \vdash d(s) : D(r_s)}{\begin{array}{c} s : T, t : T, f : \mathrm{Id}_T(s,t) \vdash j(d,f) : D(f) \\ s : T \vdash j(d, r_s) \equiv d(s) : D(r_s) \end{array}}$$

hom right elimination and computation

$$\frac{\begin{array}{c} s : T^{\text{core}}, t : T, f : \hom_T(i^{\text{op}} s, t) \vdash D(f) \ \text{type} \\ s : T^{\text{core}} \vdash d(s) : D(1_s) \end{array}}{\begin{array}{c} s : T^{\text{core}}, t : T, f : \hom_T(i^{\text{op}} s, t) \vdash e_R(d,f) : D(f) \\ s : T^{\text{core}} \vdash e_R(d, 1_s) \equiv d(s) : D(1_s) \end{array}}$$

Rules for hom: left elimination and computation

Id elimination and computation

$$\frac{s : T, t : T, f : \mathrm{Id}_T(s,t) \vdash D(f) \ \text{type} \qquad s : T \vdash d(s) : D(r_s)}{\begin{array}{c} s : T, t : T, f : \mathrm{Id}_T(s,t) \vdash j(d,f) : D(f) \\ s : T \vdash j(d, r_s) \equiv d(s) : D(r_s) \end{array}}$$

hom left elimination and computation

$$\frac{\begin{array}{c} s : T^{\mathrm{op}}, t : T^{\mathrm{core}}, f : \hom_T(s, it) \vdash D(f) \ \text{type} \\ s : T^{\mathrm{core}} \vdash d(s) : D(1_s) \end{array}}{\begin{array}{c} s : T^{\mathrm{op}}, t : T^{\mathrm{core}}, f : \hom_T(s, it) \vdash e_L(d,f) : D(f) \\ s : T^{\mathrm{core}} \vdash e_L(d, 1_s) \equiv d(s) : D(1_s) \end{array}}$$

Problems with the first attempt

The functions $\mathrm{op}, \mathrm{core}$ are problematic.

- There are no introduction rules for T^{core} or T^{op}
- Including the identity type causes the hom type to collapse to the identity type on elements of T^{core}.
- We need a op function on the universe; e.g. the **1-functor** $\mathrm{op} : \mathit{Cat} \to \mathit{Cat}$. This does not exist for 2-categories and up.

Modal directed homotopy type theory

Solution

The solution is to properly account for core, op, etc.

- **syntactically**: modal type theory
- **semantically**: multisided weak factorization systems (jww van den Berg, McCloskey)

(The theory of multisided weak factorization systems accounts for multiple fibrations – e.g. Grothendieck fibrations, opfibrations, isofibrations – in one category and how they interact, inspired by the two-sided fibrations of Street.)

Modal directed type theory

The idea:

- Forget about having a type constructors $T \mapsto T^{\mathrm{op}}, T^{\mathrm{core}}$

$$x : R^{\mathrm{op}}, y : S^{\mathrm{core}} \vdash T$$

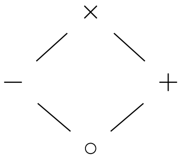
- Instead op and core should be descriptions of how variables can be used.

$$x \overset{-}{:} R, y \overset{\circ}{:} S \vdash T$$

A modal approach

Following Licata-Shulman-Riley's (2017) modal framework for the sequent calculus.

- four modes that form a lattice:



- with an multiplication
  - $+$ is the identity
  - $\circ$ is almost an absorbing element: $\circ a = a\circ = \circ$ except $\circ\times = \times$
  - $\times$ is almost an absorbing element: $\times a = a\times = \times$ except $\times\circ = \circ$
  - $-$ is a root of unity: $-- = +$.
- This is a sub-monoidal category of $[\mathrm{Cat}, \mathrm{Cat}]$

Orientations

Contexts are annotated by *orientations*. We write:

$$w \overset{\times}{:} A, x \overset{+}{:} B, y \overset{-}{:} C, z \overset{\circ}{:} D \vdash T$$

or

$$w : A, x : B, y : C, z : D \vdash_{x, y^-, z^\circ} T$$

Orientations

Contexts are annotated by *orientations*. We write:

$$w \overset{\times}{:} A, x \overset{+}{:} B, y \overset{-}{:} C, z \overset{\circ}{:} D \vdash T$$

or

$$w : A, x : B, y : C, z : D \vdash_{x, y^{-}, z^{\circ}} T$$

Orientations on contexts inherit an order from the lattice, so we use the following rule.

$$\frac{\Gamma \vdash_{\ell} \mathcal{J} \qquad m \leqslant \ell \;\; \Gamma\text{-ort}}{\Gamma \vdash_{m} \mathcal{J}} \textsc{Ort-Subst}$$

Structural rules

$$\frac{\Gamma, x : \sigma, \Delta \ \mathsf{ctx}}{\Gamma, x : \sigma, \Delta \vdash_x x : \sigma} \ \textsc{Var}$$

$$\frac{\Gamma, \Delta \vdash_{\ell,m} \mathcal{J} \qquad \Gamma \vdash_\ell \rho \ \mathsf{type}}{\Gamma, x : \rho, \Delta \vdash_{\ell,m} \mathcal{J}} \ \textsc{Weak}$$

$$\frac{\begin{array}{c}\Gamma, x : \rho, \Delta \vdash_{\ell,\omega,m} T \\ \Gamma \vdash_\ell U : \rho \qquad n \leqslant \ell \ \ \Gamma\text{-ort} \qquad n \leqslant \omega \cdot \ell \ \ \Gamma\text{-ort}\end{array}}{\Gamma, \Delta[U/x] \vdash_{n,m} T[U/x]} \ \textsc{Subst}$$

The new hom-type

hom formation

$$\frac{\Gamma \vdash_{\ell} A \text{ type}}{\Gamma, x : A, y : A \vdash_{\ell, x^{-}, y} \mathrm{hom}_A(x, y) \text{ type}} \text{ hom-}\textsc{form}$$

hom introduction

$$\frac{\Gamma \vdash_{\ell} A \text{ type}}{\Gamma, x : A \vdash_{\ell, x^{\circ}} 1_x : \mathrm{hom}_A(x, x) \text{ type}} \text{ hom-}\textsc{intro}$$

The new Id-type

Id° formation

$$\frac{\Gamma \vdash_{\ell} A \text{ type}}{\Gamma, x : A, y : A \vdash_{\ell, x^\circ, y^\circ} \mathsf{Id}^\circ_A(x, y) \text{ type}} \;\mathsf{Id}^\circ\text{-}\textsc{form}$$

Id° introduction

$$\frac{\Gamma \vdash_{\ell} A \text{ type}}{\Gamma, x : A \vdash_{\ell, x^\circ} \mathsf{refl}_x : \mathsf{Id}^\circ_A(x, x) \text{ type}} \;\mathsf{Id}^\circ\text{-}\textsc{intro}$$

The new Id-type

$\mathsf{Id}^\times$ formation

$$\frac{\Gamma \vdash_\ell A \text{ type}}{\Gamma, x : A, y : A \vdash_{\ell, x^\times, y^\times} \mathsf{Id}^\times(x, y) \text{ type}}\ \mathsf{Id}^\times\text{-}\textsc{form}$$

$\mathsf{Id}^\times$ introduction

$$\frac{\Gamma \vdash_\ell A \text{ type}}{\Gamma, x : A \vdash_{\ell, x^\times} \mathsf{refl}_x : \mathsf{Id}^\times(x, x) \text{ type}}\ \mathsf{Id}^\times\text{-}\textsc{intro}$$

Inside the type theory

What can we do?

- Find inclusions $\mathsf{Id}^{\circ}(a, b) \to \hom(a, b) \to \mathsf{Id}^{\times}(a, b)$, but not $\hom(a, b) \to \mathsf{Id}^{\circ}(a, b)$.
- Transport and compose.

What can't we do?

- Form all Σ types (F types in LSR). For example, the one you should get from $a : A \vdash_{a \vdash \mathsf{op}(a)} 1$ is A^{op}.

Future work

- Connect this formally with the intended semantics (jww van den Berg-McCloskey and Ahrens-van der Weide)
- Understand which Σ types exist.
- Π-types, directed univalence, higher inductive types, etc...

Thank you!