The univalence principle

Paige Randall North

Utrecht University

25 April 2024

Outline

1. Background on type theory and univalent foundations
2. The univalence principle[1]

[1] jww Ahrens, Shulman, Tsementzis

Outline

1. Background on type theory and univalent foundations
2. The univalence principle[2]

[2] jww Ahrens, Shulman, Tsementzis

Type theory's beginnings

- 1970s: Martin-Löf introduces his type theory
  - As a self-sufficient foundation of mathematics
  - Well-suited for machine verification

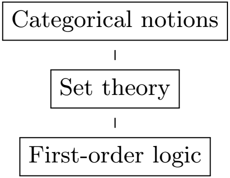
Type theory's beginnings

- 1970s: Martin-Löf introduces his type theory
  - As a **self-sufficient** foundation of mathematics
  - Well-suited for machine verification

Type theory's beginnings

- 1970s: Martin-Löf introduces his type theory
  - As a **self-sufficient** foundation of mathematics
  - Well-suited for machine verification

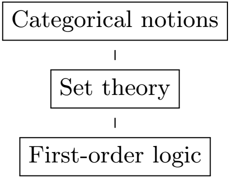
Classical mathematics:



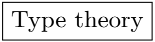
Type theory's beginnings

- 1970s: Martin-Löf introduces his type theory
  - As a **self-sufficient** foundation of mathematics
  - Well-suited for machine verification

Classical mathematics:



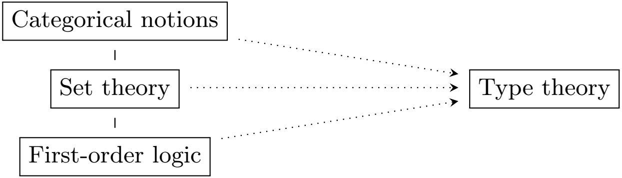
Mathematics à la Martin-Löf:



Type theory's beginnings

- 1970s: Martin-Löf introduces his type theory
  - As a **self-sufficient** foundation of mathematics
  - Well-suited for machine verification

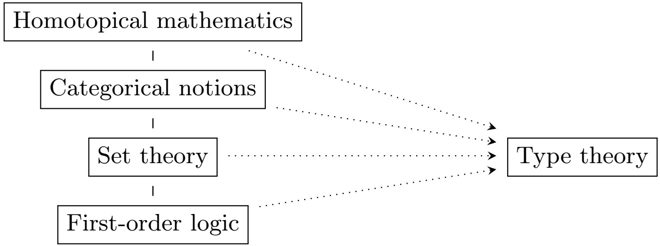
Classical mathematics: **Mathematics à la Martin-Löf:**



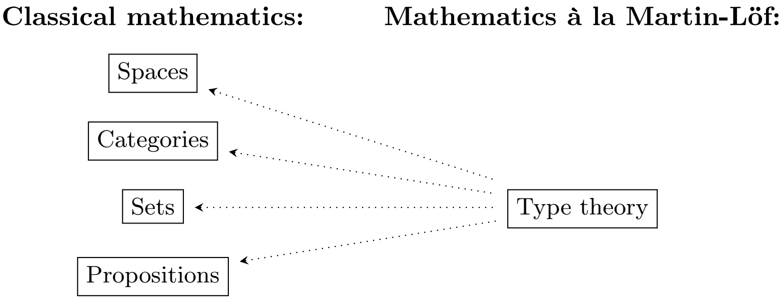
Type theory's beginnings

- 1970s: Martin-Löf introduces his type theory
  - As a **self-sufficient** foundation of mathematics
  - Well-suited for machine verification

Classical mathematics: **Mathematics à la Martin-Löf:**



Interpretations of type theory into classical mathematics



Interpretations of type theory into classical mathematics

Types	**Terms**	**Product**	**Equality**
Propositions	proofs	$\wedge$	$=$
Sets	elements	$\times$	$=$
Categories	objects	$\times$	$\cong$
Spaces	points	$\times$	$\sim$

Interpretations of type theory into classical mathematics

Types	**Terms**	**Product**	**Equality**
Propositions	proofs	$\wedge$	$=$
Sets	elements	$\times$	$=$
Categories	objects	$\times$	$\cong$
Spaces	points	$\times$	$\sim$

Interpretations of type theory into classical mathematics

Types	**Terms**	**Product**	**Equality**
Propositions	proofs	$\wedge$	$=$
Sets	elements	$\times$	$=$
Categories	objects	$\times$	$\cong$
Spaces	points	$\times$	$\sim$

- Everything in type theory respects equality

Interpretations of type theory into classical mathematics

Types	Terms	Product	Equality
Propositions	proofs	$\wedge$	$=$
Sets	elements	$\times$	$=$
Categories	objects	$\times$	$\cong$
Spaces	points	$\times$	$\sim$

- Everything in type theory respects equality
- Mathematics in type theory
  $+$ Interpretation into Xs where equality is interpreted by Y
  $\rightsquigarrow$ Mathematics in Xs up to Y

Interpretations of type theory into classical mathematics

Types	**Terms**	**Product**	**Equality**
Propositions	proofs	$\wedge$	$=$
Sets	elements	$\times$	$=$
Categories	objects	$\times$	$\cong$
Spaces	points	$\times$	$\sim$

- Everything in type theory respects equality
- Mathematics in type theory
  $+$ Interpretation into Xs where equality is interpreted by Y
  $\rightsquigarrow$ Mathematics in Xs up to Y

Interpretations of type theory into classical mathematics

Types	**Terms**	**Product**	**Equality**
Propositions	proofs	$\wedge$	$=$
Sets	elements	$\times$	$=$
Categories	objects	$\times$	$\cong$
Spaces	points	$\times$	$\sim$

- Everything in type theory respects equality
- Mathematics in type theory
  $+$ Interpretation into Xs where equality is interpreted by Y
  $\rightsquigarrow$ Mathematics in Xs up to Y

Interpretations of type theory into classical mathematics

Types	Terms	Product	Equality
Propositions	proofs	$\wedge$	$=$
Sets	elements	$\times$	$=$
Groupoids	objects	$\times$	$\cong$
Spaces	points	$\times$	$\sim$

- Everything in type theory respects equality
- Mathematics in type theory
  $+$ Interpretation into Xs where equality is interpreted by Y
  $\rightsquigarrow$ Mathematics in Xs up to Y

Different notions of equality

Synthetic vs. analytic equalities

In type theory with the equality type, we always have a ("synthetic") equality type between $a, b : D$

$$a =_D b.$$

Depending on the type D, we might also have a type of "analytic" equalities

$$a \simeq_D b.$$

A *univalence principle* for this D and this $\simeq_D$ states that

$$(a =_D b) \to (a \simeq_D b)$$

is an equivalence.

Identicals and indiscernibilites

Identity of indiscernibles

Leibniz: two things are equal when they are *indiscernible* (have the same properties).

$$(a = b) \leftarrow \big(\forall P. P(a) \leftrightarrow P(b)\big)$$

Identicals and indiscernibilites

Identity of indiscernibles

Leibniz: two things are equal when they are *indiscernible* (have the same properties).

$$(a = b) \leftrightarrow \big(\forall P. P(a) \leftrightarrow P(b)\big)$$

Identicals and indiscernibilites

Identity of indiscernibles

Leibniz: two things are equal when they are *indiscernible* (have the same properties).

$$(a =_D b) \leftrightarrow \left(\prod_{P:D\to\mathsf{Type}} P(a) \simeq P(b) \right)$$

Identicals and indiscernibilites

Identity of indiscernibles

Leibniz: two things are equal when they are *indiscernible* (have the same properties).

$$(a =_D b) \leftrightarrow \left(\prod_{P:D\to\mathsf{Type}} P(a) \simeq P(b) \right)$$

- This holds in type theory.

Identicals and indiscernibilites

Identity of indiscernibles

Leibniz: two things are equal when they are *indiscernible* (have the same properties).

$$(a =_D b) \leftrightarrow \left(\prod_{P:D \to \mathsf{Type}} P(a) \simeq P(b) \right)$$

- This holds in type theory.
- Given a univalence principle $(a =_D b) \simeq (a \simeq_D b)$, we find an *equivalence principle*:

$$(a \simeq_D b) \to \left(\prod_{P:D \to \mathsf{Type}} P(a) \simeq P(b) \right).$$

Univalence

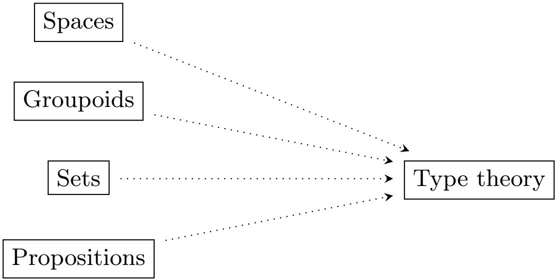
- We've seen that equality in type theory can be interpreted as notions weaker than classical equality (e.g. isomorphism, paths).
- Voevodsky imported weakness for equality from the interpretation in spaces into type theory by imposing the *Univalence Axiom* (UA):

> The canonical function $(A =_{\mathsf{Type}} B) \to (A \simeq B)$ is an equivalence of types, for any types A and B.

- UA is validated by the interpretation into spaces, but not into propositions, sets, or groupoids.
- Instead we **internalize** these notions.

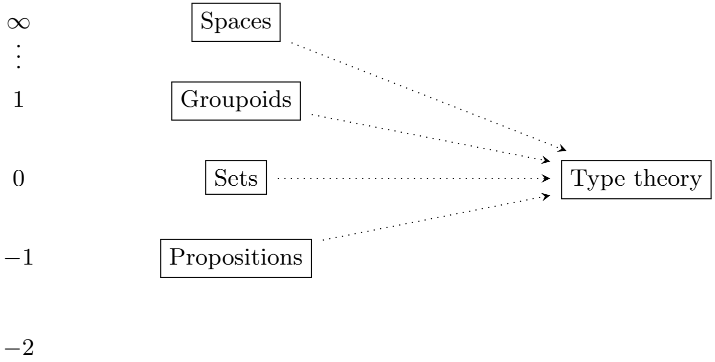
Internalization of classical mathematics into type theory

Classical mathematics: **Mathematics à la Martin-Löf:**



Internalization of classical mathematics into type theory

Classical mathematics: **Mathematics à la Martin-Löf:**



Univalent mathematics

- If two types A, B are propositions,

$$(A \simeq B) \simeq (A \leftrightarrow B)$$

Univalent mathematics

- If two types A, B are propositions,

$$(A =_{\text{Prop}} B) \overset{UA}{\simeq} (A \simeq B) \simeq (A \leftrightarrow B)$$

so everything respects bi-implication of propositions.

Univalent mathematics

- If two types A, B are propositions,

$$(A =_{\text{Prop}} B) \overset{UA}{\simeq} (A \simeq B) \simeq (A \leftrightarrow B)$$

so everything respects bi-implication of propositions.

- If A, B are sets,

$$(A =_{\text{Set}} B) \overset{UA}{\simeq} (A \simeq B) \simeq (A \cong B)$$

so everything respects bijection of sets.

Univalent mathematics

- If two types A, B are propositions,

$$(A =_{\text{Prop}} B) \overset{UA}{\simeq} (A \simeq B) \simeq (A \leftrightarrow B)$$

  so everything respects bi-implication of propositions.
- If A, B are sets,

$$(A =_{\text{Set}} B) \overset{UA}{\simeq} (A \simeq B) \simeq (A \cong B)$$

  so everything respects bijection of sets.
- For types A, B which are structured sets (groups, rings, etc),

$$(A =_{\text{Grp}} B) \overset{UA}{\simeq} (A \simeq B) \simeq (A \cong B)$$

  so everything respects isomorphism of groups (or rings, etc).[3]

[3] Coquand-Danielsson 2013

Univalent mathematics

- If two types A, B are propositions,

$$(A =_{\text{Prop}} B) \overset{UA}{\simeq} (A \simeq B) \simeq (A \leftrightarrow B)$$

  so everything respects bi-implication of propositions.
- If A, B are sets,

$$(A =_{\text{Set}} B) \overset{UA}{\simeq} (A \simeq B) \simeq (A \cong B)$$

  so everything respects bijection of sets.
- For types A, B which are structured sets (groups, rings, etc),

$$(A =_{\text{Grp}} B) \overset{UA}{\simeq} (A \simeq B) \simeq (A \cong B)$$

  so everything respects isomorphism of groups (or rings, etc).[3]
- For *univalent* categories A, B,

$$(A =_{\text{UCat}} B) \overset{UA}{\simeq} (A \simeq B) \simeq (A \simeq B)$$

  so everything respects equivalence of univalent categories.[4]

[3] Coquand-Danielsson 2013

[4] Ahrens-Kapulkin-Shulman 2015

Univalent mathematics

- Voevodsky dreamt of 'univalent mathematics' in which

$$(A =_{\mathrm{D}} B) \simeq (A \simeq_{\mathrm{D}} B)$$

  where D is any type of mathematical object (propositions, sets, groups, categories, ∞-categories, etc) and $\simeq_{\mathrm{D}}$ is the appropriate notion of 'sameness' for that type of objects.
- This would give us an appropriate language in which to study D.

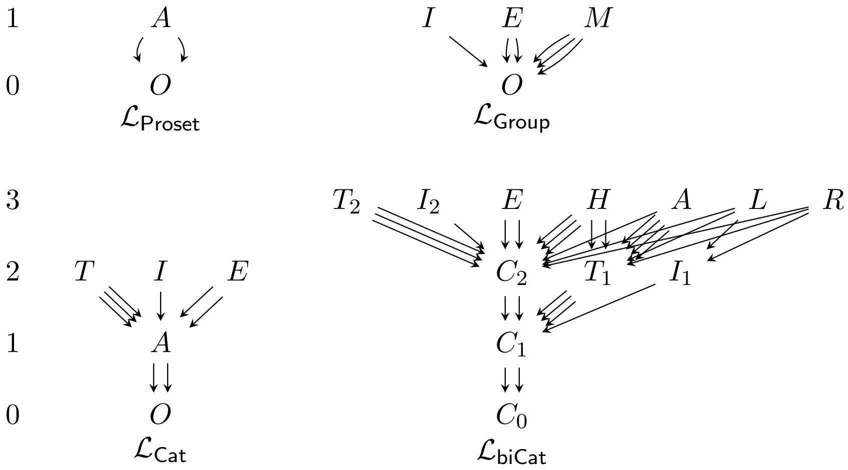
Outline

1. Background on type theory and univalent foundations

2. The univalence principle[5]

[5] jww Ahrens, Shulman, Tsementzis

Signatures


1
0
A
O
$\mathcal{L}_{\mathsf{Proset}}$
I
E
M
O
$\mathcal{L}_{\mathsf{Group}}$
3
2
1
0
T
I
E
A
O
$\mathcal{L}_{\mathsf{Cat}}$
T_2
I_2
E
H
A
L
R
C_2
T_1
I_1
C_1
C_0
$\mathcal{L}_{\mathsf{biCat}}$

Structures

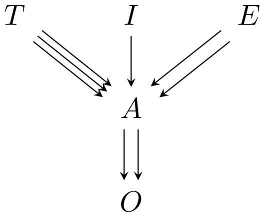
- Morally, an **$\mathcal{L}$-structure** for a signature $\mathcal{L}$ is a Reedy fibrant diagram $\mathcal{L} \to \mathsf{Type}$.

Structures

- Morally, an **$\mathcal{L}$-structure** for a signature $\mathcal{L}$ is a Reedy fibrant diagram $\mathcal{L} \to \mathsf{Type}$.
- In type theory, we define an **$\mathcal{L}$-structure** fiberwise.

Structures

- Morally, an **$\mathcal{L}$-structure** for a signature $\mathcal{L}$ is a Reedy fibrant diagram $\mathcal{L} \to \mathsf{Type}$.
- In type theory, we define an **$\mathcal{L}$-structure** fiberwise.
- An $\mathcal{L}_{\mathsf{Cat}}$-structure $\mathcal{C}$ consists of:

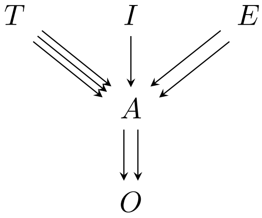


$\mathcal{L}_{\mathsf{Cat}}$

- $\mathcal{C}O : \mathsf{Type}$
- $x, y : \mathcal{C}O \vdash \mathcal{C}A(x, y) : \mathsf{Type}$
- $x : \mathcal{C}O, f : \mathcal{C}A(x, x) \vdash \mathcal{C}I_x(f) : \mathsf{Type}$
- $x, y, z : \mathcal{C}O, f : \mathcal{C}A(x, y), g : \mathcal{C}A(y, z), h : \mathcal{C}A(x, z) \vdash \mathcal{C}T_{x,y,z}(f, g, h) : \mathsf{Type}$
- $x, y : \mathcal{C}O, f, g : \mathcal{C}A(x, y) \vdash \mathcal{C}E_{x,y}(f, g) : \mathsf{Type}$

Structures

- Morally, an **$\mathcal{L}$-structure** for a signature $\mathcal{L}$ is a Reedy fibrant diagram $\mathcal{L} \to \mathsf{Type}$.
- In type theory, we define an **$\mathcal{L}$-structure** fiberwise.
- An $\mathcal{L}_{\mathsf{Cat}}$-structure $\mathcal{C}$ consists of:



$\mathcal{L}_{\mathsf{Cat}}$

- $\mathcal{C}O : \mathsf{Type}$
- $x, y : \mathcal{C}O \vdash \mathcal{C}A(x, y) : \mathsf{Type}$
- $x : \mathcal{C}O, f : \mathcal{C}A(x, x) \vdash \mathcal{C}I_x(f) : \mathsf{Type}$
- $x, y, z : \mathcal{C}O, f : \mathcal{C}A(x, y), g : \mathcal{C}A(y, z), h : \mathcal{C}A(x, z) \vdash \mathcal{C}T_{x,y,z}(f, g, h) : \mathsf{Type}$
- $x, y : \mathcal{C}O, f, g : \mathcal{C}A(x, y) \vdash \mathcal{C}E_{x,y}(f, g) : \mathsf{Type}$

- Then we add axioms.

Level-wise equivalence

Proposition

For two $\mathcal{L}$-structures S, T,

$$(S =_{\mathcal{L}-\mathsf{Str}} T) \simeq (S \cong_{\mathcal{L}-\mathsf{Str}} T)$$

where $\cong_{\mathcal{L}-\mathsf{Str}}$ denotes levelwise equivalence.

Level-wise equivalence

Proposition

For two $\mathcal{L}$-structures S, T,

$$(S =_{\mathcal{L}-\mathsf{Str}} T) \simeq (S \cong_{\mathcal{L}-\mathsf{Str}} T)$$

where $\cong_{\mathcal{L}-\mathsf{Str}}$ denotes levelwise equivalence.

A levelwise equivalence $\mathcal{C} \cong_{\mathcal{L}_{\mathsf{Cat}}-\mathsf{Str}} \mathcal{D}$ consists of:

- $e_O : \mathcal{C}O \xrightarrow{\sim} \mathcal{D}O$
- $x, y : \mathcal{C}O \vdash e_A : \mathcal{C}A(x, y) \xrightarrow{\sim} \mathcal{D}(e_O x, e_O y)$
- $x : \mathcal{C}O, f : \mathcal{C}A(x, x) \vdash e_I : \mathcal{C}I_x(f) \xrightarrow{\sim} \mathcal{D}I_{e_O x}(e_A f)$
- $x, y, z : \mathcal{C}O, f : \mathcal{C}A(x, y), g : \mathcal{C}A(y, z), h : \mathcal{C}A(x, z) \vdash$
  $\mathcal{C}T_{x,y,z}(f, g, h) \xrightarrow{\sim} \mathcal{D}T_{e_O x, e_O y, e_O z}(e_A f, e_A g, e_A h)$
- $x, y : \mathcal{C}O, f, g : \mathcal{C}A(x, y) \vdash \mathcal{C}E_{x,y}(f, g) \xrightarrow{\sim} \mathcal{C}E_{e_O x, e_O y}(e_A f, e_A g)$

Level-wise equivalence

Proposition

For two $\mathcal{L}$-structures S, T,

$$(S =_{\mathcal{L}-\mathsf{Str}} T) \simeq (S \cong_{\mathcal{L}-\mathsf{Str}} T)$$

where $\cong_{\mathcal{L}-\mathsf{Str}}$ denotes levelwise equivalence.

A levelwise equivalence $\mathcal{C} \cong_{\mathcal{L}_{\mathsf{Cat}}-\mathsf{Str}} \mathcal{D}$ consists of:

- $e_O : \mathcal{C}O \xrightarrow{\sim} \mathcal{D}O$
- $x, y : \mathcal{C}O \vdash e_A : \mathcal{C}A(x, y) \xrightarrow{\sim} \mathcal{D}(e_O x, e_O y)$
- $x : \mathcal{C}O, f : \mathcal{C}A(x, x) \vdash e_I : \mathcal{C}I_x(f) \xrightarrow{\sim} \mathcal{D}I_{e_O x}(e_A f)$
- $x, y, z : \mathcal{C}O, f : \mathcal{C}A(x, y), g : \mathcal{C}A(y, z), h : \mathcal{C}A(x, z) \vdash$ $\mathcal{C}T_{x,y,z}(f, g, h) \xrightarrow{\sim} \mathcal{D}T_{e_O x, e_O y, e_O z}(e_A f, e_A g, e_A h)$
- $x, y : \mathcal{C}O, f, g : \mathcal{C}A(x, y) \vdash \mathcal{C}E_{x,y}(f, g) \xrightarrow{\sim} \mathcal{C}E_{e_O x, e_O y}(e_A f, e_A g)$

But this is not an equivalence of categories.

Level-wise equivalence

Proposition

For two $\mathcal{L}$-structures S, T,

$$(S =_{\mathcal{L}-\mathsf{Str}} T) \simeq (S \cong_{\mathcal{L}-\mathsf{Str}} T)$$

where $\cong_{\mathcal{L}-\mathsf{Str}}$ denotes levelwise equivalence.

A levelwise equivalence $\mathcal{C} \cong_{\mathcal{L}_{\mathsf{Cat}}-\mathsf{Str}} \mathcal{D}$ consists of:

- $e_O : \mathcal{C}O \xrightarrow{\sim} \mathcal{D}O$
- $x, y : \mathcal{C}O \vdash e_A : \mathcal{C}A(x, y) \xrightarrow{\sim} \mathcal{D}(e_O x, e_O y)$
- $x : \mathcal{C}O, f : \mathcal{C}A(x, x) \vdash e_I : \mathcal{C}I_x(f) \xrightarrow{\sim} \mathcal{D}I_{e_O x}(e_A f)$
- $x, y, z : \mathcal{C}O, f : \mathcal{C}A(x, y), g : \mathcal{C}A(y, z), h : \mathcal{C}A(x, z) \vdash$ $\mathcal{C}T_{x,y,z}(f, g, h) \xrightarrow{\sim} \mathcal{D}T_{e_O x, e_O y, e_O z}(e_A f, e_A g, e_A h)$
- $x, y : \mathcal{C}O, f, g : \mathcal{C}A(x, y) \vdash \mathcal{C}E_{x,y}(f, g) \xrightarrow{\sim} \mathcal{C}E_{e_O x, e_O y}(e_A f, e_A g)$

But this is not an equivalence of categories.
And is it appropriate to call $\mathcal{C}, \mathcal{D}$ categories?

Indiscernibility

Definition

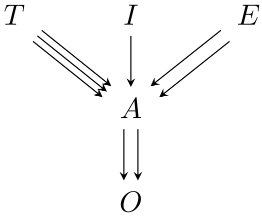
Given an $\mathcal{L}$-structure M, and an object S of $\mathcal{L}$, we say that two elements $x, y : MS$ are *indiscernible* if substituting x for y in any object of $\mathcal{L}$ that depends on (i.e. object with a morphism to) S produces equivalent types.

Definition

An $\mathcal{L}$-structure M is *univalent* if for any object S of $\mathcal{L}$, and any $x, y : MS$, the type of indiscernibilities between x and y is equivalent to the type of equalities between x and y.

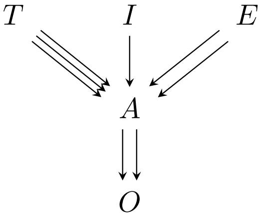
Univalent $\mathcal{L}_{\text{cat}}$ structures

Let $\mathcal{C}$ be a univalent $\mathcal{L}_{\text{cat}}$ structure.



Univalent $\mathcal{L}_{\text{cat}}$ structures

Let $\mathcal{C}$ be a univalent $\mathcal{L}_{\text{cat}}$ structure.



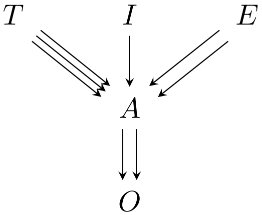
- Any two terms $x : \mathcal{C}O, f : \mathcal{C}A(x,x) \vdash i, j : \mathcal{C}I_x(f)$ are indiscernible.

→ Each $\mathcal{C}I_x(f)$ is a proposition.

→ Similarly, each $\mathcal{C}T_{x,y,z}(f,g,h)$, $\mathcal{C}E_{x,y}(f,g)$ is a proposition.

Univalent $\mathcal{L}_{\text{cat}}$ structures

Let $\mathcal{C}$ be a univalent $\mathcal{L}_{\text{cat}}$ structure.



- Any two terms $x : \mathcal{C}O, f : \mathcal{C}A(x,x) \vdash i, j : \mathcal{C}I_x(f)$ are indiscernible.

→ Each $\mathcal{C}I_x(f)$ is a proposition.

→ Similarly, each $\mathcal{C}T_{x,y,z}(f,g,h)$, $\mathcal{C}E_{x,y}(f,g)$ is a proposition.

- In the axioms for a category, we have that E behaves like equality (is reflexive and a congruence for T, I, E.)

→ Univalence at A means that $f = g$ is equivalent to $\mathcal{C}E_{x,y}(f,g)$.

→ $\mathcal{C}A(x,y)$ is a set.

Univalent $\mathcal{L}_{\mathrm{cat}}$ structures

- The indiscernibilities between $a, b : \mathcal{C}O$ consist of
  - $\phi_{x\bullet} : \mathcal{C}A(x,a) \cong \mathcal{C}A(x,b)$ for each $x : \mathcal{C}O$
  - $\phi_{\bullet z} : \mathcal{C}A(a,z) \cong \mathcal{C}A(b,z)$ for each $z : \mathcal{C}O$
  - $\phi_{\bullet\bullet} : \mathcal{C}A(a,a) \cong \mathcal{C}A(b,b)$
  - The following for all appropriate w, x, y, z, f, g, h:

$$\mathcal{C}T_{x,y,a}(f,g,h) \leftrightarrow \mathcal{C}T_{x,y,b}(f, \phi_{y\bullet}(g), \phi_{x\bullet}(h))$$

$$\mathcal{C}T_{x,a,z}(f,g,h) \leftrightarrow \mathcal{C}T_{x,b,z}(\phi_{x\bullet}(f), \phi_{\bullet z}(g), h)$$

$$\mathcal{C}T_{a,z,w}(f,g,h) \leftrightarrow \mathcal{C}T_{b,z,w}(\phi_{\bullet z}(f), g, \phi_{\bullet w}(h))$$

$$\mathcal{C}T_{x,a,a}(f,g,h) \leftrightarrow \mathcal{C}T_{x,b,b}(\phi_{x\bullet}(f), \phi_{\bullet\bullet}(g), \phi_{x\bullet}(h))$$

$$\mathcal{C}T_{a,x,a}(f,g,h) \leftrightarrow \mathcal{C}T_{b,x,b}(\phi_{\bullet x}(f), \phi_{x\bullet}(g), \phi_{\bullet\bullet}(h))$$

$$\mathcal{C}T_{a,a,x}(f,g,h) \leftrightarrow \mathcal{C}T_{b,b,x}(\phi_{\bullet\bullet}(f), \phi_{\bullet x}(g), \phi_{\bullet x}(h))$$

$$\mathcal{C}T_{a,a,a}(f,g,h) \leftrightarrow \mathcal{C}T_{b,b,b}(\phi_{\bullet\bullet}(f), \phi_{\bullet\bullet}(g), \phi_{\bullet\bullet}(h))$$

$$\mathcal{C}I_a(f) \leftrightarrow \mathcal{C}I_b(\phi_{\bullet\bullet}(f))$$

$$\mathcal{C}E_{x,a}(f,g) \leftrightarrow \mathcal{C}E_{x,b}(\phi_{x\bullet}(f), \phi_{x\bullet}(g$$

$$\mathcal{C}E_{a,x}(f,g) \leftrightarrow \mathcal{C}E_{b,x}(\phi_{\bullet x}(f), \phi_{\bullet x}(g$$

$$\mathcal{C}E_{a,a}(f,g) \leftrightarrow \mathcal{C}E_{b,b}(\phi_{\bullet\bullet}(f), \phi_{\bullet\bullet}(g$$

Univalent $\mathcal{L}_{\text{cat}}$ structures

- The indiscernibilities between $a, b : \mathcal{C}O$ consist of
  - $\phi_{x\bullet} : \mathcal{C}A(x,a) \cong \mathcal{C}A(x,b)$ for each $x : \mathcal{C}O$
  - $\phi_{\bullet z} : \mathcal{C}A(a,z) \cong \mathcal{C}A(b,z)$ for each $z : \mathcal{C}O$
  - $\phi_{\bullet\bullet} : \mathcal{C}A(a,a) \cong \mathcal{C}A(b,b)$
  - The following for all appropriate w, x, y, z, f, g, h:

$$\mathcal{C}T_{x,y,a}(f,g,h) \leftrightarrow \mathcal{C}T_{x,y,b}(f, \phi_{y\bullet}(g), \phi_{x\bullet}(h))$$
$$\mathcal{C}T_{x,a,z}(f,g,h) \leftrightarrow \mathcal{C}T_{x,b,z}(\phi_{x\bullet}(f), \phi_{\bullet z}(g), h)$$
$$\mathcal{C}T_{a,z,w}(f,g,h) \leftrightarrow \mathcal{C}T_{b,z,w}(\phi_{\bullet z}(f), g, \phi_{\bullet w}(h))$$
$$\mathcal{C}T_{x,a,a}(f,g,h) \leftrightarrow \mathcal{C}T_{x,b,b}(\phi_{x\bullet}(f), \phi_{\bullet\bullet}(g), \phi_{x\bullet}(h))$$
$$\mathcal{C}T_{a,x,a}(f,g,h) \leftrightarrow \mathcal{C}T_{b,x,b}(\phi_{\bullet x}(f), \phi_{x\bullet}(g), \phi_{\bullet\bullet}(h))$$
$$\mathcal{C}T_{a,a,x}(f,g,h) \leftrightarrow \mathcal{C}T_{b,b,x}(\phi_{\bullet\bullet}(f), \phi_{\bullet x}(g), \phi_{\bullet x}(h))$$
$$\mathcal{C}T_{a,a,a}(f,g,h) \leftrightarrow \mathcal{C}T_{b,b,b}(\phi_{\bullet\bullet}(f), \phi_{\bullet\bullet}(g), \phi_{\bullet\bullet}(h))$$

$$\mathcal{C}I_a(f) \leftrightarrow \mathcal{C}I_b(\phi_{\bullet\bullet}(f))$$
$$\mathcal{C}E_{x,a}(f,g) \leftrightarrow \mathcal{C}E_{x,b}(\phi_{x\bullet}(f), \phi_{x\bullet}(g$$
$$\mathcal{C}E_{a,x}(f,g) \leftrightarrow \mathcal{C}E_{b,x}(\phi_{\bullet x}(f), \phi_{\bullet x}(g$$
$$\mathcal{C}E_{a,a}(f,g) \leftrightarrow \mathcal{C}E_{b,b}(\phi_{\bullet\bullet}(f), \phi_{\bullet\bullet}(g$$

- But this an isomorphism in the usual categorical sense.

→ Univalence at O means that $x = y$ is equivalent to $x \cong y$.

The right notion of equivalence

Main theorem

For two *univalent* $\mathcal{L}$-structures S, T,

$$(S =_{\mathcal{L}-\mathsf{Str}} T) \simeq (S \cong_{\mathcal{L}-\mathsf{Str}} T) \simeq (S \cong^*_{\mathcal{L}-\mathsf{Str}} T) \simeq (S \twoheadrightarrow T)$$

where $\cong^*_{\mathcal{L}-\mathsf{Str}}$ denotes levelwise equivalence up to indiscernbility and $\twoheadrightarrow$ denotes a very split surjective morphism.

The right notion of equivalence

Main theorem

For two *univalent* $\mathcal{L}$-structures S, T,

$$(S =_{\mathcal{L}-\mathsf{Str}} T) \simeq (S \cong_{\mathcal{L}-\mathsf{Str}} T) \simeq (S \cong^*_{\mathcal{L}-\mathsf{Str}} T) \simeq (S \twoheadrightarrow T)$$

where $\cong^*_{\mathcal{L}-\mathsf{Str}}$ denotes levelwise equivalence up to indiscernbility and $\twoheadrightarrow$ denotes a very split surjective morphism.

Very surjective morphisms of $\mathcal{L}_{\text{cat}}$-structures

A *very surjective morphism* or *equivalence* $F : \mathcal{C} \simeq \mathcal{D}$ of $\mathcal{L}_{\text{cat}}$-structures consists of surjections

- $FO : \mathcal{C}O \twoheadrightarrow \mathcal{D}O$
- $FA : \mathcal{C}A(x, y) \twoheadrightarrow \mathcal{D}A(Fx, Fy)$ for every $x, y : \mathcal{C}O$
- $FT : \mathcal{C}T(f, g, h) \twoheadrightarrow \mathcal{D}T(Ff, Fg, Fh)$ for all $f : \mathcal{C}A(x, y), g : \mathcal{C}A(y, z), h : \mathcal{C}A(x, z)$
- $FE : \mathcal{C}E(f, g) \twoheadrightarrow \mathcal{D}E(Ff, Fg)$ for all $f, g : \mathcal{C}A(x, y)$
- $FI : \mathcal{C}I(f) \twoheadrightarrow \mathcal{D}I(Ff)$ for all $f : \mathcal{C}A(x, x)$

The right notion of equivalence

Main theorem

For two *univalent* $\mathcal{L}$-structures S, T,

$$(S =_{\mathcal{L}-\mathsf{Str}} T) \simeq (S \cong_{\mathcal{L}-\mathsf{Str}} T) \simeq (S \cong^*_{\mathcal{L}-\mathsf{Str}} T) \simeq (S \twoheadrightarrow T)$$

where $\cong^*_{\mathcal{L}-\mathsf{Str}}$ denotes levelwise equivalence up to indiscernbility and $\twoheadrightarrow$ denotes a very split surjective morphism.

Very surjective morphisms of $\mathcal{L}_{\text{cat}}$-structures

A *very surjective morphism* or *equivalence* $F : \mathcal{C} \simeq \mathcal{D}$ of univalent $\mathcal{L}_{\text{cat}}$-structures consists of surjections

- $FO : \mathcal{C}O \twoheadrightarrow \mathcal{D}O$
- $FA : \mathcal{C}A(x, y) \twoheadrightarrow \mathcal{D}A(Fx, Fy)$ for every $x, y : \mathcal{C}O$
- $FT : \mathcal{C}T(f, g, h) \twoheadrightarrow \mathcal{D}T(Ff, Fg, Fh)$ for all $f : \mathcal{C}A(x, y), g : \mathcal{C}A(y, z), h : \mathcal{C}A(x, z)$
- $FE : \mathcal{C}E(f, g) \twoheadrightarrow \mathcal{D}E(Ff, Fg)$ for all $f, g : \mathcal{C}A(x, y)$
- $FI : \mathcal{C}I(f) \twoheadrightarrow \mathcal{D}I(Ff)$ for all $f : \mathcal{C}A(x, x)$

The right notion of equivalence

Main theorem

For two *univalent* $\mathcal{L}$-structures S, T,

$$(S =_{\mathcal{L}-\mathsf{Str}} T) \simeq (S \cong_{\mathcal{L}-\mathsf{Str}} T) \simeq (S \cong^*_{\mathcal{L}-\mathsf{Str}} T) \simeq (S \twoheadrightarrow T)$$

where $\cong^*_{\mathcal{L}-\mathsf{Str}}$ denotes levelwise equivalence up to indiscernbility and $\twoheadrightarrow$ denotes a very split surjective morphism.

Very surjective morphisms of $\mathcal{L}_{\text{cat}}$-structures

A *very surjective morphism* or *equivalence* $F : \mathcal{C} \simeq \mathcal{D}$ of univalent $\mathcal{L}_{\text{cat}}$-structures consists of surjections

- $FO : \mathcal{C}O \twoheadrightarrow \mathcal{D}O$
- $FA : \mathcal{C}A(x, y) \twoheadrightarrow \mathcal{D}A(Fx, Fy)$ for every $x, y : \mathcal{C}O$
- $FT : \mathcal{C}T(f, g, h) \leftrightarrow \mathcal{D}T(Ff, Fg, Fh)$ for all $f : \mathcal{C}A(x, y), g : \mathcal{C}A(y, z), h : \mathcal{C}A(x, z)$
- $FE : \mathcal{C}E(f, g) \leftrightarrow \mathcal{D}E(Ff, Fg)$ for all $f, g : \mathcal{C}A(x, y)$
- $FI : \mathcal{C}I(f) \leftrightarrow \mathcal{D}I(Ff)$ for all $f : \mathcal{C}A(x, x)$

The right notion of equivalence

Main theorem

For two *univalent* $\mathcal{L}$-structures S, T,

$$(S =_{\mathcal{L}-\mathsf{Str}} T) \simeq (S \cong_{\mathcal{L}-\mathsf{Str}} T) \simeq (S \cong^*_{\mathcal{L}-\mathsf{Str}} T) \simeq (S \twoheadrightarrow T)$$

where $\cong^*_{\mathcal{L}-\mathsf{Str}}$ denotes levelwise equivalence up to indiscernbility and $\twoheadrightarrow$ denotes a very split surjective morphism.

Very surjective morphisms of $\mathcal{L}_{\text{cat}}$-structures

A *very surjective morphism* or *equivalence* $F : \mathcal{C} \simeq \mathcal{D}$ of univalent $\mathcal{L}_{\text{cat}}$-structures consists of surjections

- $FO : \mathcal{C}O \twoheadrightarrow \mathcal{D}O$
- $FA : \mathcal{C}A(x, y) \twoheadrightarrow \mathcal{D}A(Fx, Fy)$ for every $x, y : \mathcal{C}O$
- $FT : \mathcal{C}T(f, g, h) \leftrightarrow \mathcal{D}T(Ff, Fg, Fh)$ for all $f : \mathcal{C}A(x, y), g : \mathcal{C}A(y, z), h : \mathcal{C}A(x, z)$
- $FE : (f = g) \leftrightarrow (Ff = Fg)$ for all $f, g : \mathcal{C}A(x, y)$
- $FI : \mathcal{C}I(f) \leftrightarrow \mathcal{D}I(Ff)$ for all $f : \mathcal{C}A(x, x)$

The right notion of equivalence

Main theorem

For two *univalent* $\mathcal{L}$-structures S, T,

$$(S =_{\mathcal{L}-\mathsf{Str}} T) \simeq (S \cong_{\mathcal{L}-\mathsf{Str}} T) \simeq (S \cong^*_{\mathcal{L}-\mathsf{Str}} T) \simeq (S \twoheadrightarrow T)$$

where $\cong^*_{\mathcal{L}-\mathsf{Str}}$ denotes levelwise equivalence up to indiscernbility and $\twoheadrightarrow$ denotes a very split surjective morphism.

Very surjective morphisms of $\mathcal{L}_{\text{cat}}$-structures

A *very surjective morphism* or *equivalence* $F : \mathcal{C} \simeq \mathcal{D}$ of univalent $\mathcal{L}_{\text{cat}}$-structures consists of surjections

- $FO : \mathcal{C}O \twoheadrightarrow \mathcal{D}O$
- $FA : \mathcal{C}A(x, y) \cong \mathcal{D}A(Fx, Fy)$ for every $x, y : \mathcal{C}O$
- $FT : \mathcal{C}T(f, g, h) \leftrightarrow \mathcal{D}T(Ff, Fg, Fh)$ for all $f : \mathcal{C}A(x, y), g : \mathcal{C}A(y, z), h : \mathcal{C}A(x, z)$
- $FE : (f = g) \leftrightarrow (Ff = Fg)$ for all $f, g : \mathcal{C}A(x, y)$
- $FI : \mathcal{C}I(f) \leftrightarrow \mathcal{D}I(Ff)$ for all $f : \mathcal{C}A(x, x)$

Summary

For every signature $\mathcal{L}$, we have

- a notion of structure,
- a notion of indiscernibility within each sort,
- a notion of univalent structures,
- a notion of equivalence,
- a univalence theorem.

Summary

For every signature $\mathcal{L}$, we have

- a notion of structure,
- a notion of indiscernibility within each sort,
- a notion of univalent structures,
- a notion of equivalence,
- a univalence theorem.

The paper includes examples of

- †-categories,
- profunctors,
- bicategories,
- opetopic bicategories,
- ...

Current and future work

- Drop the splitness condition for certain structures.
- Extend to infinite structures.
- Formulate an analogue to the Rezk completion.
- Translate the theory into one about structures which can include explicit functions.
- Explore mathematics within examples.
- Give a model-category-theoretic account.

Thank you!