Univalent foundations and the equivalence principle

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Outline

The equivalence principle

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Reasoning in mathematics should be **invariant under** the appropriate notion of **equivalence**.

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Reasoning in mathematics should be **invariant under** the appropriate notion of **equivalence**.

Notion of equivalence depends on the objects under consideration:

- **equal** numbers, functions,. . .
- **isomorphic** sets, groups, rings,. . .
- **equivalent** categories
- **biequivalent** bicategories
- . . .

Non-examples: statements violating equivalence principle

We can easily **violate** this principle:

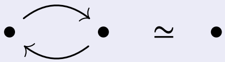
Exercise

Find a statement about sets that is not invariant under isomorphism:

$$\{\emptyset, \{\emptyset\}\} \cong \{\emptyset, \{\{\emptyset\}\}\}$$

Exercise

Find a statement about categories that is not invariant under equivalence:



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$\{\emptyset\} \in X$

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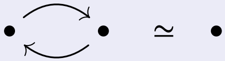

$\mathscr{C}$ has exactly 1 object.

A language for invariant properties

Michael Makkai, *Towards a Categorical Foundation of Mathematics*:

> *"The basic character of the Principle of Isomorphism is that of a* ***constraint on the language*** *of Abstract Mathematics; a welcome one, since it provides for the separation of sense from nonsense."*

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Goal

To have a **syntactic criterion** for properties and constructions that are invariant under equivalence

How to break the equivalence principle for categories. . .

- Recall: the statement

  The category $\mathscr{C}$ has exactly one object.

  is not invariant under equivalence of categories.
- In general, referring to **equality of objects** breaks invariance, but. . .

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- even the **definition** of category refers to equality of objects:

Problem

"If $\mathrm{dom}(g)$ is **equal to** $\mathrm{cod}(f)$, then $g \circ f$ exists."

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Problem

"If $\mathrm{dom}(g)$ is **equal to** $\mathrm{cod}(f)$, then $g \circ f$ exists."

Can we give a definition of category without using equality of objects?

. . . and how to fix it.

Solution

Use a logic/language of **dependent sets**, in which $\mathrm{dom}(g) = \mathrm{cod}(f)$ is encoded by what type of thing f and g are.

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A category consists of

- a set O of objects
- for each $x, y \in O$, a type/set $A(x, y)$ of arrows
- for each $x, y, z \in O$ and each $f \in A(x, y)$ and $g \in A(y, z)$, a type/set $g \circ f \in A(x, z)$
- for each $x \in O$, an identity $\mathrm{id}_x \in A(x, x)$
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Gives rise to **dependently typed language** by adding logical connectors.

Invariance for statements

Theorem (Freyd '76, Blanc '78)

A property of categories (expressed in 2-sorted first order logic) is invariant under equivalence iff it can be expressed in this dependently typed language, using equality for arrows but not for objects.

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- What about **constructions** on categories?

Invariance for statements

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- What about **constructions** on categories?
- What about other mathematical structures?

Outline

Overview of types in Martin-Löf type theory

Type former	Notation	canonical term
Dependent type	$x : A \vdash B(x)$	
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Boolean type	Bool	$\top, \bot$
Natural numbers type	Nat	$0, sx$
Sum type	$\sum_{x:A} B(x)$	(a, b)
Product type	$\prod_{x:A} B(x)$	$\lambda(x : A).b$
Identity type	$x : A, y : A \vdash x = y$	$\mathrm{refl}(x) : x = x$
Universe	Type	

Curry-Howard Correspondence

We can interpret these types as propositions or sets.

Properties of the identity type

Induction principle for $a = b$

To define a function

$$f : \prod_{(x,y:A)} \prod_{(p:x=y)} C(x,y,p)$$

it suffices to specify its image on $(x, x, \mathrm{refl} x)$.

- sym : $\prod_{x,y:A}(x = y) \to (y = x)$
- trans : $\prod_{x,y,z:A}(x = y) \times (y = z) \to (x = z)$

The equality principle in type theory

Any predicate or construction that can be defined on terms of a type A is of the form $f : A \to B$.

- The predicate "G is an abelian group" is a function $Grp \to Prop$.
- Considering the lattice of subgroups of any group G produces a function $Grp \to Latt$.

Equality principle

$$\prod_{x,y:A} (x = y) \to \prod_{f:A \to B} \big(f(x) = f(y)\big)$$

Space interpretation

The identity type behaves like equality:

- reflexivity, symmetry, transitivity
- Everything respects equality

but more like paths in a space:

- Can iterate identity type
- Cannot show that any two identities are identical

Voevodsky Correspondence

We can interpret

- a type K as a *Kan complex* $[K]$
- a dependent type $x : B \vdash E(b)$ as a *Kan fibration* $[p] : [E] \to [B]$
- a dependent term $x : B \vdash e(b) : E(b)$ as a *section* of $[e]$ of $[p]$
- a term $p : a \to_K b$ as a path from a to b in K

The Univalence Axiom

There are two notions of 'sameness' between types:

- $A = B$
- $A \simeq B$ (functions $f : A \leftrightarrows B : g$ such that $fg = 1$ and $gf = 1$)

There is always a function

$$(A = B) \to (A \simeq B)$$

which is an equivalence in Kan complexes.

The Univalence Axiom

The function

$$(A = B) \to (A \simeq B)$$

is an equivalence.

This is true in Kan complexes.

Outline

Strategy

We always have a version of the equivalence principle:

Equality principle

$$\prod_{x,y:A}(x = y) \to \prod_{f:A\to B}\big(f(x) = f(y)\big)$$

but we want better ones where we replace the 'synthetic' equality $x = y$ with an 'analytic' equality $x \cong y$ which depends on the type.

Strategy: prove that the function $(x = y) \to (x \cong y)$ is an equivalence

Univalence principle

$(x =_T y) \cong (x \cong_T y)$

for a type T and appropriate $\cong_T$. Then we will get:

Equivalence principle

$$\prod_{x,y:A}(x \cong y) \to \prod_{f:A\to B}\big(f(x) = f(y)\big)$$

Contractible types, propositions and sets

- A is **contractible**

$$\mathsf{isContr}(A) \;:\equiv\; \sum_{x:A}\prod_{y:A} y = x$$

- A is a **proposition**

$$\mathsf{isProp}(A) \;:\equiv\; \prod_{x,y:A} x = y$$

- A is a **set**

$$\mathsf{isSet}(A) \;:\equiv\; \prod_{x,y:A} \mathsf{isProp}(x = y)$$

$$\mathsf{Prop} \;:\equiv\; \sum_{X:\mathsf{Type}} \mathsf{isProp}(X) \qquad \mathsf{Set} \;:\equiv\; \sum_{X:\mathsf{Type}} \mathsf{isSet}(X)$$

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Univalence for Propositions and Sets

Immediate consequences of the univalence axiom:

Univalence for propositions

$P =_{\mathsf{Prop}} Q \;\simeq\; P \leftrightarrow Q$

Univalence for sets

$P =_{\mathsf{Set}} Q \;\simeq\; P \cong Q$

Monoids in type theory

In type theory, a monoid is a tuple $(M, \mu, e, \alpha, \lambda, \rho)$ where

1. $M : \mathbf{Set}$
2. $\mu : M \times M \to M$
3. $e : M$
4. $\alpha : \Pi_{(a,b,c:M)} \mu(\mu(a,b),c) = \mu(a,\mu(b,c))$
5. $\lambda : \Pi_{(a:M)} \mu(e,a) = a$
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Why M : **Set**?

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Why $M : \mathbf{Set}$?

Abstractly, a monoid is a (dependent) pair $(\mathit{data}, \mathit{proof})$ where

- *data* is 1.–3.
- *proof* is 4.–6.

Structure Identity Principle

Univalence for monoids

$$M =_{\mathsf{Monoid}} N \quad \simeq \quad M \cong N$$

We also have univalence for other set-level strucuters (Coquand-Danielsson):

- groups, rings
- posets
- discrete fields
- sets with fixpoint operator

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What about **categories**?

Univalence for categories

We only have univalence for **univalent** categories: ones where the canonical function $A = B \to A \cong B$ for objects $A, B : \mathscr{C}$ is an equivalence.

Here, the homsets are sets, and the type of objects will be groupoids.

Univalence for univalent categories

$$\mathscr{C} =_{\mathsf{UCat}} \mathscr{D} \;\simeq\; C \simeq D$$

We also have univalence for other higher strucuters (Ahrens-North-Shulman-Tsementzis):

- bicategories, tricategories, etc
- double categories
- dagger categories

Further resources

- HoTT Reading Group, 10:30-12 on Wednesdays
- HoTT Book
  - https://homotopytypetheory.org/book/
- Learn how to write proofs in a computer!
  - https://leanprover-community.github.io/learn.html
  - (Number Game)

Thank you!