A type theory for directed homotopy theory

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Outline

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- Directed paths are introduced as terms of a type former, hom, to be added to Martin-Löf type theory
- Transport along terms of hom
- Independence of hom and Id

What does directed mean?

Syntactically

Martin-Löf's Id type is symmetric/undirected since for any type T, and terms $a, b : T$, there is a function

$$i : \mathrm{Id}_T(a, b) \to \mathrm{Id}_T(b, a)$$

so that any *path* $p : \mathrm{Id}_T(a, b)$ can be *inverted* to obtain a path $ip : \mathrm{Id}_T(b, a)$.

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- Can think of these terms as *undirected* paths
- Can we design a type former of *directed* paths that resembles Id but without its inversion operation i?

What does directed mean?

Semantically

higher groupoids

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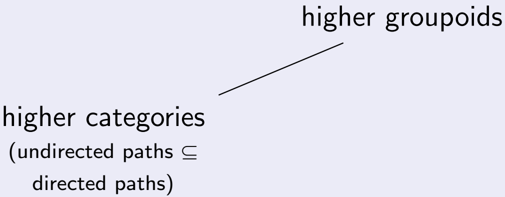
Semantically



What does directed mean?

Semantically

higher groupoids

higher categories
(undirected paths ⊆ directed paths)

directed spaces
(directed paths ⊆ undirected paths)

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Directed spaces

Rough definition

A (topological) space together with a subset of its paths that are marked as ‘directed’

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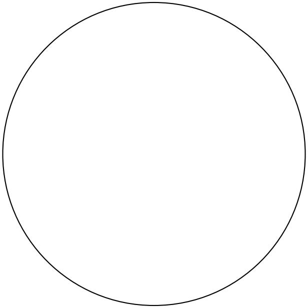
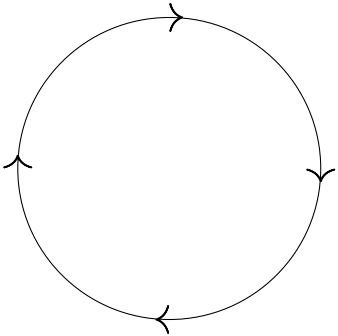


Directed spaces

Rough definition

A (topological) space together with a subset of its paths that are marked as 'directed'



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Concurrent processes can be represented by directed spaces.

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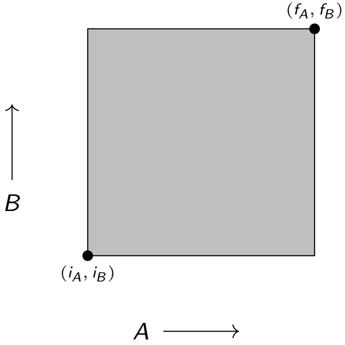

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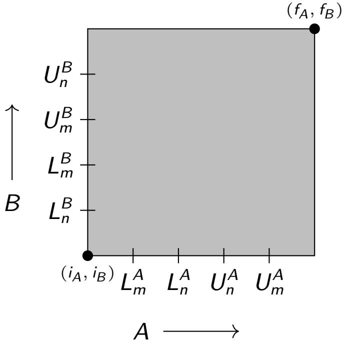

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- m, n are two memory locations
- which can be locked (L) or unlocked (U) by each process

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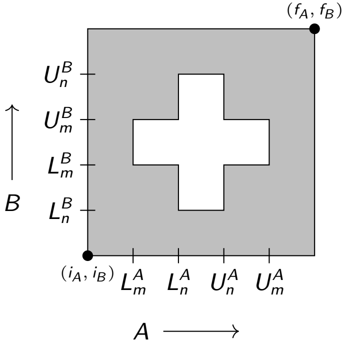

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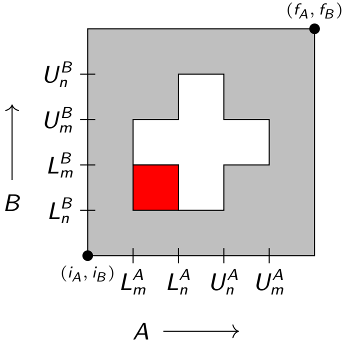

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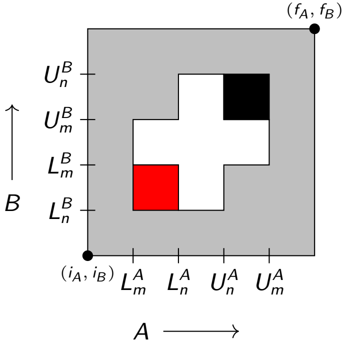

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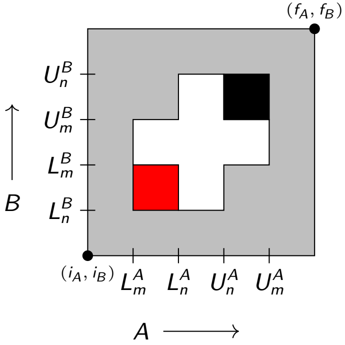

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Fundamental questions:

- Which states are safe?
- Which states are reachable?

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Rules for hom: core and op

$$\frac{T \ \textsc{type}}{T^{\mathsf{core}} \ \textsc{type}}$$

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Id introduction

$$\frac{T \ \ \textsc{type} \qquad t : T}{r_t : \mathrm{Id}_T(t, t) \ \ \textsc{type}}$$

Rules for hom: right elimination and computation

$$\frac{T \;\; \text{TYPE} \qquad s : T^{\text{core}}, t : T, f : \hom_T(i^{\text{op}} s, t) \vdash D(f) \;\; \text{TYPE} \qquad s : T^{\text{core}} \vdash d(s) : D(1_s)}{\begin{array}{c} s : T^{\text{core}}, t : T, f : \hom_T(i^{\text{op}} s, t) \vdash e_R(d, f) : D(f) \\ s : T^{\text{core}} \vdash e_R(d, 1_s) \equiv d(s) : D(1_s) \end{array}}$$

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Syntactic results

- Transport: for a dependent type $t : T \vdash S(t)$:

$$t : T^{\text{core}}, t' : T, f : \hom_T(i^{\text{op}} t, t'), s : S(it) \\ \vdash \text{transport}_{\text{R}}(s, f) : S(t')$$

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- Composition: for a type T:

$$r : T^{\mathrm{op}}, s : T^{\mathrm{core}}, t : T, f : \mathrm{hom}_T(r, is), g : \mathrm{hom}_T(i^{\mathrm{op}}s, t) \\ \vdash \mathrm{comp}_{\mathrm{R}}(f, g) : \mathrm{hom}_T(r, t)$$

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  and the introduction rule just gives $1_t : hom_T(t, t)$.
- Plugs into the homotopy theory of morphisms, as the interpretation of the Id type plugs into the homotopy theory of isomorphisms.

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  - integrate Id and hom in the same theory
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Future work

We need to:

- integrate this into traditional Martin-Löf type theory
  - integrate Id and hom in the same theory
  - specify Σ, Π, etc
- find interpretations in categories of directed spaces
  - build 'directed' weak factorization systems
  - build universes

Summary & future work

The future

We aim to define and reason about

$$isReachable(T) := \Sigma_{x:T} \hom_T(i, x)$$

$$isSafe(T) := \Sigma_{x:T^{op}} \hom_T(x, f)$$

for any type T with terms $i : T^{op}, f : T$.

Thank you!

The interpretation

- Use the framework of comprehension categories
- Dependent types are represented by functors $T : \Gamma \to Cat$.
- Dependent terms are represented by natural transformations

  $$\Gamma \underset{T}{\overset{*}{\rightrightarrows}} Cat \quad \Downarrow t$$

  where $* : \Gamma \to Cat$ is the functor which takes everything to the one-object category.
- Context extension is represented by the Grothendieck construction which takes each functor $T : \Gamma \to Cat$ to the Grothendieck opfibration

  $$\pi_\Gamma : \int_\Gamma T \to \Gamma.$$

Interpreting core and op in the empty context

$$\frac{T \ \text{TYPE}}{T^{\text{core}} \ \text{TYPE} \qquad T^{\text{op}} \ \text{TYPE}} \qquad\qquad \frac{T \ \text{TYPE} \qquad t : T^{\text{core}}}{it : T \qquad i^{\text{op}}t : T^{\text{op}}}$$

For any category T,

- $T^{\text{core}} := \text{ob}(T)$
- $T^{\text{op}} := T^{\text{op}}$
- $i : T^{\text{core}} \to T$ and $i^{\text{op}} : T^{\text{core}} \to T^{\text{op}}$ are the identity on objects.

Interpreting hom formation and introduction

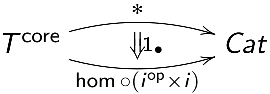
$$\frac{T \ \ \text{TYPE} \qquad s : T^{\mathrm{op}} \qquad t : T}{\hom_T(s, t) \ \ \text{TYPE}} \qquad\qquad \frac{T \ \ \text{TYPE} \qquad t : T^{\mathrm{core}}}{1_t : \hom_T(i^{\mathrm{op}} t, it) \ \ \text{TYPE}}$$

For any category T,

- Take the functor

$$\hom : T^{\mathrm{op}} \times T \to \mathit{Set} \hookrightarrow \mathit{Cat}.$$

- Take the natural transformation



where each component $1_t : * \to \hom(t, t)$ picks out the identity morphism of t.

Interpreting right hom elimination and computation

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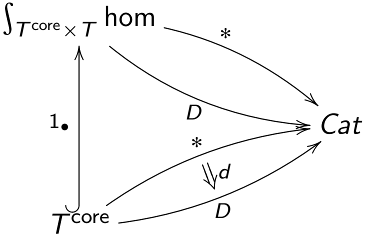

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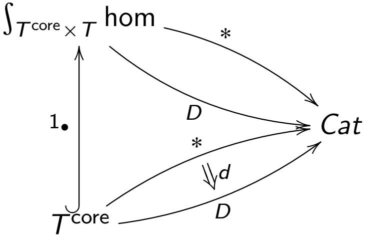

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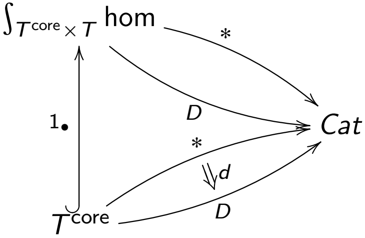

- Use the fact that the subcategory T^{core} is 'initial':
  - for every $(s, t, f) \in \int_{T^{\text{core}} \times T} \hom$ there is a unique morphism $(1_s, f) : (s, s, 1_s) \to (s, t, f)$ with domain in T^{core}

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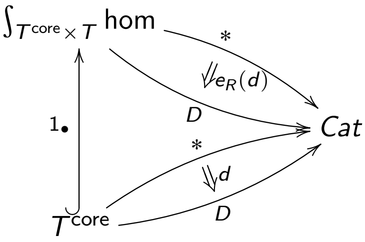

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- Set $e_R(d)_{(s,t,f)} := D(1_s, f) d_{(s,s,1_s)}$

Interpreting left hom elimination and computation

$$\frac{T \;\; \text{TYPE} \qquad s : T^{\mathrm{op}}, t : T^{\mathrm{core}}, f : \hom_T(s, it) \vdash D(f) \;\; \text{TYPE} \qquad s : T^{\mathrm{core}} \vdash d(s) : D(1_s)}{\begin{array}{c} s : T^{\mathrm{op}}, t : T^{\mathrm{core}}, f : \hom_T(s, it) \vdash e_L(d, f) : D(f) \\ s : T^{\mathrm{core}} \vdash e_L(d, 1_s) \equiv d(s) : D(1_s) \end{array}}$$

- Replace T by T^{op} and apply right hom elimination and computation.

A homotopical perspective

While the homotopy theory of isomorphisms in categories

$$\mathcal{C} \to \mathcal{C}^{(\cong)} \to \mathcal{C} \times \mathcal{C}$$

provides an interpretation of Martin-Löf's identity type, the homotopy theory of morphisms in categories

$$\mathcal{C} \to \mathcal{C}^{(\to)} \to \mathcal{C} \times \mathcal{C}$$

provides an interpretation of this hom former.

The weak factorization system

- Let $(\cong)$ denote the category with two objects and one isomorphism between them.
- Let $(\rightarrow)$ denote the category with two objects and one morphism between them.
- Then factorize the codiagonal of the one-point category in two ways

$$* + * \;\rightarrow\; (\cong) \;\rightarrow\; * \qquad\qquad * + * \;\rightarrow\; (\rightarrow) \;\rightarrow\; *$$

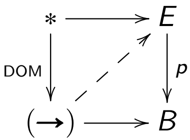
- which produces a factorization of any diagonal in two ways which each generate weak factorization systems.

$$\mathcal{C} \rightarrow \mathcal{C}^{(\cong)} \rightarrow \mathcal{C} \times \mathcal{C} \qquad\qquad \mathcal{C} \rightarrow \mathcal{C}^{(\rightarrow)} \rightarrow \mathcal{C} \times \mathcal{C}$$

- The first gives an interpretation of the Id type in *Cat*.
- The second underlies this interpretation of the hom type in *Cat*.

The weak factorization system continued

- The right class of this weak factorization system are those functors $p : E \to B$ which have the enriched lifting property



- so all Grothendieck opfibrations (dependent projections) are in the right class.
- The functor $1_\bullet : T^{\text{core}} \hookrightarrow \int_{T^{\text{core}} \times T} \text{hom}$ is the left part of the factorization of

$$i : T^{\text{core}} \to T.$$

- Then the right hom elimination and computation rule arises from the weak factorization system.