

Coinductive control of inductive data types

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Published in CALCO 2023
arXiv:2303.16793

15 June 2023

Outline

Introduction and background

Endofunctors

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Overview

Theorem (N.-Péroux)

The category of algebras over an *accessible, lax symmetric monoidal* endofunctor on a *locally presentable, symmetric monoidal closed* category is enriched over the category of coalgebras of the same endofunctor.

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Examples

There are many examples, including containers¹ with extra structure.

¹Recall Stefania Damato's talk

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Gain

Get more control over algebras

- ▶ Get more “initial algebras” (e.g. generalized W-types)

¹Recall Stefania Damato’s talk

Review of categorical W-types

Let \mathcal{C} be a locally presentable, symmetric monoidal closed category, i.e. \mathbf{Set} .

Natural numbers

The type of natural numbers \mathbb{N} is the initial algebra for the endofunctor $X \mapsto X + 1$.

This endofunctor fulfills our hypotheses.

Lists

The type of lists $\mathbb{List}(A)$ is the initial algebra for the endofunctor $X \mapsto X \times A + 1$.

When A is equipped with the structure of a commutative monoid, this fulfills our hypotheses.

Previous work on coalgebraic enrichment

Universal measuring coalgebra (Wraith, Sweedler 1968)

For k -algebras A and B , there is a k -coalgebra $\underline{\text{Alg}}(A, B)$

- ▶ which underlies an enrichment of k -algebras in k -coalgebras
- ▶ whose *set-like elements*² are in bijection with $\text{Alg}(A, B)$.

Taking $B := k$, one gets the *dual* $\underline{\text{Alg}}(A, k)$ of A .

Extensions

- ▶ Anel-Joyal 2013 (dg-algebras)
- ▶ Hyland-Franco-Vasilakopoulou 2017 (monoids)
- ▶ Vasilakopoulou 2019 (\mathcal{V} -categories)
- ▶ Péroux 2022 (∞ -algebras of an ∞ -operad)
- ▶ McDermott-Rivas-Uustalu 2022 (monads)
- ▶ N-Péroux 2023 (algebras of endofunctor)

²those $c \in \underline{\text{Alg}}(A, B)$ s.t. $\Delta c = c \otimes c$ and $\epsilon(c) = 1_A$

Enriched categories³

Definition

An *enrichment* of a category \mathcal{C} in a monoidal category \mathcal{V} consists of

- ▶ a functor $\underline{\mathcal{C}}(-, -) : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{V}$
- ▶ a morphism $\mathbb{1} \rightarrow \underline{\mathcal{C}}(A, A)$ for each object A of \mathcal{C}
- ▶ a morphism $\underline{\mathcal{C}}(A, B) \otimes \underline{\mathcal{C}}(B, C) \rightarrow \underline{\mathcal{C}}(A, C)$ for each triple A, B, C of objects of \mathcal{C}
- ▶ an isomorphism $\mathcal{V}(\mathbb{1}, \underline{\mathcal{C}}(A, B)) \cong \mathcal{C}(A, B)$

Remark

Monoidal *closed* means enriched in itself.

³Recall Niels van der Weide's talk

Outline

Introduction and background

Endofunctors

Measuring in general

Fix a category \mathcal{C} and an endofunctor F satisfying our hypotheses.

Measuring

For algebras $(A, \alpha), (B, \beta)$ a *measure* $(A, \alpha) \rightarrow (B, \beta)$ is a coalgebra (C, χ) together with a morphism $\phi : C \rightarrow \underline{\mathcal{C}}(A, B)$ such that

$$\begin{array}{ccccccc} C & \xrightarrow{\chi} & FC & \xrightarrow{F(\phi)} & F(\underline{\mathcal{C}}(A, B)) & \longrightarrow & \underline{\mathcal{C}}(FA, FB) \\ & \searrow \phi & & & & & \downarrow \beta \\ & & \underline{\mathcal{C}}(A, B) & \xrightarrow{\alpha} & \underline{\mathcal{C}}(FA, B) & & \end{array}$$

i.e., the measure and the co/algebra structures are compatible.

The *universal measure* $\underline{\text{Alg}} A, B$ is the terminal measure $(A, \alpha) \rightarrow (B, \beta)$.

Measuring for the natural numbers

Consider the endofunctor $X \mapsto X + 1$ on Set .

- ▶ Algebras are sets A together with $A + 1 \rightarrow A$
 - ▶ Have $-_A : \mathbb{N} \rightarrow A$
- ▶ Coalgebras are sets C together with $A \rightarrow A + 1$
 - ▶ Have $\llbracket - \rrbracket : C \rightarrow \mathbb{N}^\infty$

Measuring

For algebras A, B , a *measure* $A \rightarrow B$ is a coalgebra C together with a function $C \rightarrow A \rightarrow B$ such that

- ▶ $f_c(0_A) = 0_B$ for all $c \in C$;
- ▶ $f_c(a + 1) = 0_B$ for all $\llbracket c \rrbracket = 0$ and for all $a \in A$;
- ▶ $f_c(a + 1) = f_{c-1}(a) + 1$ for $\llbracket c \rrbracket \geq 1$ and for all $a \in A$.

The *universal measure* $\underline{\text{Alg}}A, B$ is the terminal measure $A \rightarrow B$.

Set-like elements in general

Definition

The *set-like elements* are

$$\mathbb{1} \rightarrow \underline{\text{Alg}}(A, B)$$

i.e., elements of $\text{Alg}(A, B)$.

Set-like elements for the natural numbers

Set-like elements

The *set-like elements* are

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Set-like elements for the natural numbers

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where \mathbb{N} has underlying set $\{*\}$ such that $\llbracket * \rrbracket = \infty$

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$$\mathbb{I} \rightarrow \underline{\text{Alg}}(A, B)$$

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so $\mathbb{I} \rightarrow \underline{\text{Alg}}(A, B)$ is an element $c \in \underline{\text{Alg}}(A, B)$ s.t. $\llbracket c \rrbracket = \infty$

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so f_c is an algebra homomorphism

Measuring

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- ▶ ...
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Example

$$\text{Alg}(\mathbb{N}, A) \cong *$$

$$\underline{\text{Alg}}(\mathbb{N}, A) \cong \mathbb{N}^\infty$$

What are the non-set-like elements?

Example

$$\text{Alg}(\mathbb{N}, A) \cong *$$

$$\underline{\text{Alg}}(\mathbb{N}, A) \cong \mathbb{N}^\infty$$

The elements corresponding to $n \in \mathbb{N}^\infty$ are functions which 'are algebra homomorphisms' on $\{0, \dots, n\} \subseteq \mathbb{N}$, i.e., are *n-partial homomorphisms*.

- ▶ Let \mathfrak{n} denote the quotient of \mathbb{N} by $m = n$ for all $m \geq n$.
- ▶ Let \mathfrak{n}° denote the subobject of \mathbb{N}^∞ consisting of $\{0, \dots, n\}$.

Example

$$\text{Alg}(\mathfrak{n}, A) \cong \begin{cases} \mathbb{N}^\infty & \text{if } n_A = m_A \text{ for all } m \geq n; \\ \mathfrak{n}^\circ & \text{otherwise.} \end{cases}$$

What can we do with this?

Perhaps define more general *initial objects*.

C -initial objects

For a coalgebra C , a C -initial algebra is an algebra A universal with the property that for all other algebras B there is a unique

$$C \rightarrow \underline{\text{Alg}}(A, B).$$

Examples

For the natural-numbers endofunctor:

- ▶ \mathbb{N} is the I -initial algebra
- ▶ \mathbb{N} is the \mathbb{N}^∞ -initial algebra
- ▶ \mathfrak{n} is the \mathfrak{n}° -initial algebra

Future work

- ▶ Work out more examples in detail
- ▶ Understand what it means to endow the containers with extra structure (e.g. A needs a commutative monoid structure for the container for $\mathbb{L}ist(A)$)
- ▶ Understand C -initial algebras in more examples and in general
- ▶ Understand if this extra structure is useful for programming languages

Thank you!