An introduction to univalent foundations and the univalence principle

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Utrecht University

7 February 2023

Outline

1. Background on type theory and univalent foundations

2. The univalence principle[1]

[1] jww Ahrens, Shulman, Tsementzis

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1. Background on type theory and univalent foundations

2. The univalence principle[2]

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Type theory's beginnings

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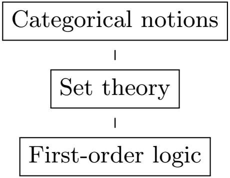

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Classical mathematics:

Categorical notions

Set theory

First-order logic

Mathematics à la Martin-Löf:

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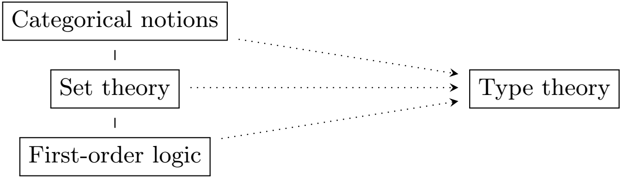

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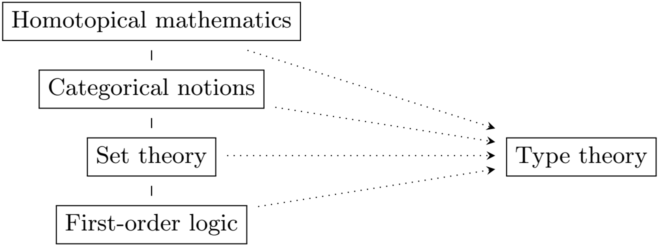
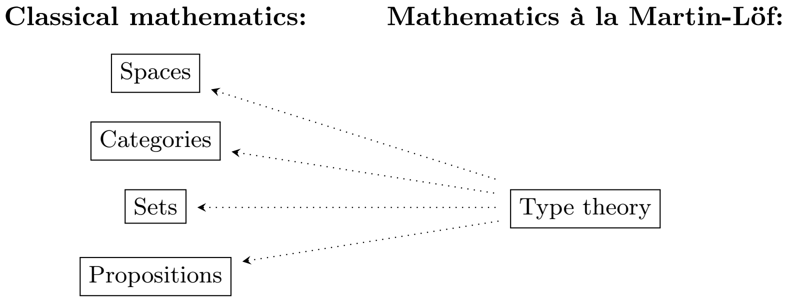


What does type theory look like?

- In mathematics, statements look like the following:
  - Consider a natural number n. The sum $n + n$ is even.
  - Consider a space X. The cone on X is contractible.
- In type theory, we write this as
  - $n : \mathbb{N} \vdash e(n) : \mathsf{isEven}(n + n)$
  - $X : \mathsf{Spaces} \vdash c(X) : \mathsf{isContr}(CX)$
- Type theory provides:
  - natural numbers type $\mathbb{N}$
  - product type $A \times B$
  - sum type $A + B$
  - function type $A \to B$
  - a universe type Type
  - a type (!) of equalities $a =_A b$
  - etc

Interpretations of type theory into classical mathematics



Interpretations of type theory into classical mathematics

Types	**Terms**	**Product**	**Equality**
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Sets	elements	$\times$	$=$
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Different notions of equality

Synthetic vs. analytic equalities

In type theory with the equality type, we always have a ("synthetic") equality type between $a, b : D$

$$a =_D b.$$

Depending on the type D, we might also have a type of "analytic" equalities

$$a \simeq_D b.$$

A *univalence principle* for this D and this $\simeq_D$ states that

$$(a =_D b) \to (a \simeq_D b)$$

is an equivalence.

Identicals and indiscernibilites

Identity of indiscernibles

Leibniz: two things are equal when they are *indiscernible* (have the same properties).

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- This holds in type theory.
- Given a univalence principle $(a =_D b) \simeq (a \simeq_D b)$, we find an *equivalence principle*:

$$(a \simeq_D b) \to \left(\prod_{P:D \to \mathsf{Type}} P(a) \simeq P(b) \right).$$

Univalence

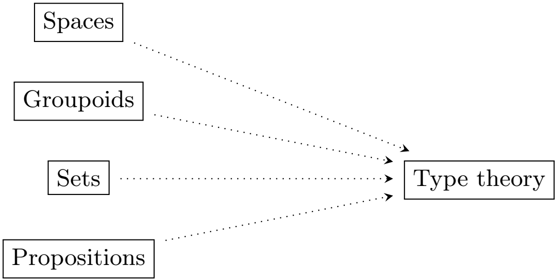
- We've seen that equality in type theory can be interpreted as notions weaker than classical equality (e.g. isomorphism, paths).
- Voevodsky imported weakness for equality from the interpretation in spaces into type theory by imposing the *Univalence Axiom* (UA):

The canonical function $(A =_{\mathsf{Type}} B) \to (A \simeq B)$ is an equivalence of types, for any types A and B.

- UA is validated by the interpretation into spaces, but not into propositions, sets, or groupoids.
- Instead we **internalize** these notions.

Internalization of classical mathematics into type theory

Classical mathematics: **Mathematics à la Martin-Löf:**



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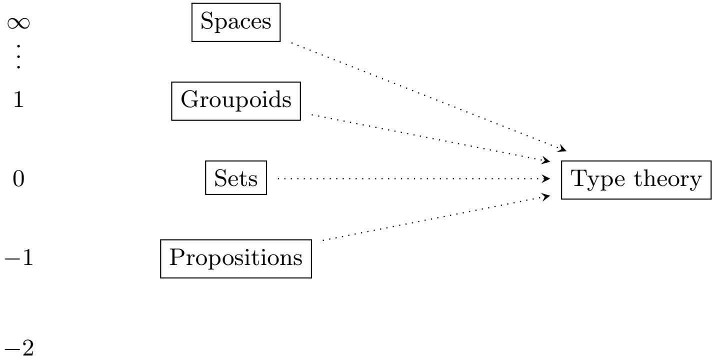

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- For types A, B which are structured sets (groups, rings, etc),

$$(A =_{\mathrm{Grp}} B) \overset{UA}{\simeq} (A \simeq B) \simeq (A \cong B)$$

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- For *univalent* categories A, B,

$$(A =_{\text{UCat}} B) \overset{UA}{\simeq} (A \simeq B) \simeq (A \simeq B)$$

  so everything respects equivalence of univalent categories.[4]

[3] Coquand-Danielsson 2013

[4] Ahrens-Kapulkin-Shulman 2015

Univalent mathematics

- Voevodsky dreamt of ‘univalent mathematics’ in which

$$(A =_{\mathrm{D}} B) \simeq (A \simeq_{\mathrm{D}} B)$$

  where D is any type of mathematical object (propositions, sets, groups, categories, ∞-categories, etc) and $\simeq_{\mathrm{D}}$ is the appropriate notion of ‘sameness’ for that type of objects.
- This would give us an appropriate language in which to study D.

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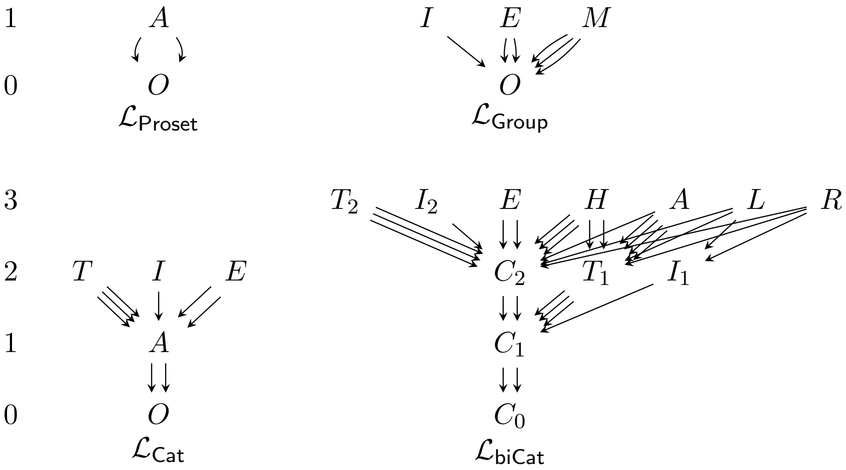
Signatures


1
0
A
O
$\mathcal{L}_{\mathsf{Proset}}$
I E M
O
$\mathcal{L}_{\mathsf{Group}}$
3
2
1
0
T I E
A
O
$\mathcal{L}_{\mathsf{Cat}}$
T_2 I_2 E H A L R
C_2 T_1 I_1
C_1
C_0
$\mathcal{L}_{\mathsf{biCat}}$

Structures

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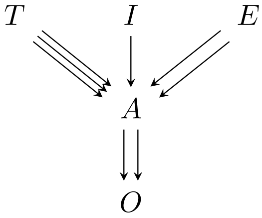

$\mathcal{L}_{\mathsf{Cat}}$

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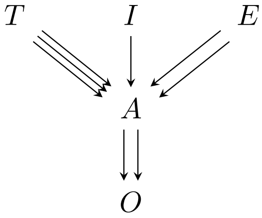

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- Then we add axioms.

Level-wise equivalence

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For two $\mathcal{L}$-structures S, T,

$$(S =_{\mathcal{L}-\mathsf{Str}} T) \simeq (S \cong_{\mathcal{L}-\mathsf{Str}} T)$$

where $\cong_{\mathcal{L}-\mathsf{Str}}$ denotes levelwise equivalence.

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But this is not an equivalence of categories.
And is it appropriate to call $\mathcal{C}, \mathcal{D}$ categories?

Indiscernibility

Definition

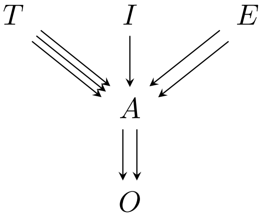
Given an $\mathcal{L}$-structure M, and an object S of $\mathcal{L}$, we say that two elements $x, y : MS$ are *indiscernible* if substituting x for y in any object of $\mathcal{L}$ that depends on (i.e. object with a morphism to) S produces equivalent types.

Definition

An $\mathcal{L}$-structure M is *univalent* if for any object S of $\mathcal{L}$, and any $x, y : MS$, the type of indiscernibilities between x and y is equivalent to the type of equalities between x and y.

Univalent $\mathcal{L}_{\text{cat}}$ structures

Let $\mathcal{C}$ be a univalent $\mathcal{L}_{\text{cat}}$ structure.



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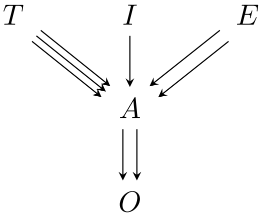

- Any two terms $x : \mathcal{C}O, f : \mathcal{C}A(x, x) \vdash i, j : \mathcal{C}I_x(f)$ are indiscernible.

→ Each $\mathcal{C}I_x(f)$ is a proposition.

→ Similarly, each $\mathcal{C}T_{x,y,z}(f, g, h)$, $\mathcal{C}E_{x,y}(f, g)$ is a proposition.

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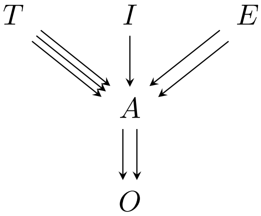

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- In the axioms for a category, we have that E behaves like equality (is reflexive and a congruence for T, I, E.)

→ Univalence at A means that $f = g$ is equivalent to $\mathcal{C}E_{x,y}(f, g)$.

→ $\mathcal{C}A(x, y)$ is a set.

Univalent $\mathcal{L}_{\text{cat}}$ structures

- The indiscernibilities between $a, b : \mathcal{C}O$ consist of
  - $\phi_{x\bullet} : \mathcal{C}A(x, a) \cong \mathcal{C}A(x, b)$ for each $x : \mathcal{C}O$
  - $\phi_{\bullet z} : \mathcal{C}A(a, z) \cong \mathcal{C}A(b, z)$ for each $z : \mathcal{C}O$
  - $\phi_{\bullet\bullet} : \mathcal{C}A(a, a) \cong \mathcal{C}A(b, b)$
  - The following for all appropriate w, x, y, z, f, g, h:

$$\mathcal{C}T_{x,y,a}(f, g, h) \leftrightarrow \mathcal{C}T_{x,y,b}(f, \phi_{y\bullet}(g), \phi_{x\bullet}(h))$$
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Univalent $\mathcal{L}_{\text{cat}}$ structures

- The indiscernibilities between $a, b : \mathcal{C}O$ consist of
  - $\phi_{x\bullet} : \mathcal{C}A(x,a) \cong \mathcal{C}A(x,b)$ for each $x : \mathcal{C}O$
  - $\phi_{\bullet z} : \mathcal{C}A(a,z) \cong \mathcal{C}A(b,z)$ for each $z : \mathcal{C}O$
  - $\phi_{\bullet\bullet} : \mathcal{C}A(a,a) \cong \mathcal{C}A(b,b)$
  - The following for all appropriate w, x, y, z, f, g, h:

$$\mathcal{C}T_{x,y,a}(f,g,h) \leftrightarrow \mathcal{C}T_{x,y,b}(f, \phi_{y\bullet}(g), \phi_{x\bullet}(h))$$
$$\mathcal{C}T_{x,a,z}(f,g,h) \leftrightarrow \mathcal{C}T_{x,b,z}(\phi_{x\bullet}(f), \phi_{\bullet z}(g), h)$$
$$\mathcal{C}T_{a,z,w}(f,g,h) \leftrightarrow \mathcal{C}T_{b,z,w}(\phi_{\bullet z}(f), g, \phi_{\bullet w}(h))$$
$$\mathcal{C}T_{x,a,a}(f,g,h) \leftrightarrow \mathcal{C}T_{x,b,b}(\phi_{x\bullet}(f), \phi_{\bullet\bullet}(g), \phi_{x\bullet}(h))$$
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$$\mathcal{C}I_a(f) \leftrightarrow \mathcal{C}I_b(\phi_{\bullet\bullet}(f))$$
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- But this an isomorphism in the usual categorical sense.

$\rightarrow$ Univalence at O means that $x = y$ is equivalent to $x \cong y$.

The right notion of equivalence

Main theorem

For two *univalent* $\mathcal{L}$-structures S, T,

$$(S =_{\mathcal{L}-\mathsf{Str}} T) \simeq (S \cong_{\mathcal{L}-\mathsf{Str}} T) \simeq (S \cong^*_{\mathcal{L}-\mathsf{Str}} T) \simeq (S \twoheadrightarrow T)$$

where $\cong^*_{\mathcal{L}-\mathsf{Str}}$ denotes levelwise equivalence up to indiscernbility and $\twoheadrightarrow$ denotes a very split surjective morphism.

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Very surjective morphisms of $\mathcal{L}_{\mathrm{cat}}$-structures

A *very surjective morphism* or *equivalence* $F : \mathcal{C} \simeq \mathcal{D}$ of $\mathcal{L}_{\mathrm{cat}}$-structures consists of surjections

- $FO : \mathcal{C}O \twoheadrightarrow \mathcal{D}O$
- $FA : \mathcal{C}A(x, y) \twoheadrightarrow \mathcal{D}A(Fx, Fy)$ for every $x, y : \mathcal{C}O$
- $FT : \mathcal{C}T(f, g, h) \twoheadrightarrow \mathcal{D}T(Ff, Fg, Fh)$ for all $f : \mathcal{C}A(x, y), g : \mathcal{C}A(y, z), h : \mathcal{C}A(x, z)$
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Summary

For every signature $\mathcal{L}$, we have

- a notion of structure,
- a notion of indiscernibility within each sort,
- a notion of univalent structures,
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The paper includes examples of

- †-categories,
- profunctors,
- bicategories,
- opetopic bicategories,
- ...

Current and future work

- Drop the splitness condition for certain structures.
- Extend to infinite structures.
- Formulate an analogue to the Rezk completion.
- Translate the theory into one about structures which can include explicit functions.
- Explore mathematics within examples.
- Give a model-category-theoretic account.

Thank you!