Seminar Logic and Foundations of Computing
Homework 1

Sara Rousta

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Problem 1. Let $\mathcal{D}$ be a nonempty small category.

(a) (5 pt) Show, using the Yoneda Lemma, that for every $d \in \operatorname{obj} \mathcal{D}$ the colimit of representables $\mathcal{D}(d, -)$ is the one-point set.

(b) (5 pt) Use (a) to show $2 \implies 3$ of Lemma 2.13:

Let $F : \mathcal{D}' \to \mathcal{D}$ be a functor such that it satisfies the finality condition with respect to all representable functors $\mathcal{D}(d, -)$. Show that for every object d of $\mathcal{D}$, the slice category $d \downarrow F$ is connected.

(Compare to Exercises 4 and 5 of IX.3, Final Functors in Categories for the Working Mathematician by Mac Lane.)

Solution.

(a) Consider the following natural isomorphisms:

$$\begin{aligned}
\mathbf{Set}(\operatorname{colim} \mathcal{D}(d, -), X) &\cong [\mathcal{D}, \mathbf{Set}](\mathcal{D}(d, -), \Delta X) \\
&\cong \Delta X(d) \\
&= X \\
&\cong \mathbf{Set}(1, X),
\end{aligned}$$

where these are consequences of the Yoneda Lemma. Again, by a corollary to Yoneda, it follows that $\operatorname{colim} \mathcal{D}(d, -) \cong 1$. $\square$

(b) Fix d of $\mathcal{D}$. Note that it follows by assumption and part (a) that the canonical morphism $\operatorname{colim} \mathcal{D}(d, F-) \to 1$ is an isomorphism. Thus, the slice category $d \downarrow F$ is nonempty. Indeed, $d \downarrow F$ is nonempty whenever $\mathcal{D}'$ is nonempty. If $\mathcal{D}'$ were empty we would have that the colimit of the empty category, being the initial object, is isomorphic to 1 in **Set**, a contradiction.

To see that for every pair of objects d', d'' of $\mathcal{D}'$, there exists a zigzag of morphisms connecting $d \to Fd'$ and $d \to Fd''$ in $d \downarrow F$, recall how colimits in **Set** look like:

$$\operatorname{colim} \mathcal{D}(d, F-) = \left(\sum_{d \in \operatorname{ob} \mathcal{D}'} \mathcal{D}(d, Fd') \right) / \sim,$$

where the quivalence relation is generated by $(d \to Fd') \sim \mathcal{D}(d, Ff)(d \to Fd')$ for all $d' \xrightarrow{f} d''$ of $\mathcal{D}'$ and $d \to Fd'$ in $\mathcal{D}(d, Ff)$. We have that $(d \to Fd') \sim (d \to Fd'')$ precisely when there is a zigzag of morphisms connecting them, and that $\operatorname{colim} \mathcal{D}(d, F-) \cong 1$. It follows that there must exist a zigzag for every pair of objects in $d \downarrow F$. $\square$