Seminar Logic and Foundations of Computing
Model Solution of Homework 2

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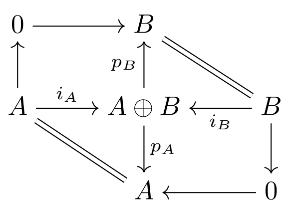
An object 0 that is both initial and terminal is called a *zero object*. In a category with a zero object, a *biproduct* of objects A and B is an object $A \oplus B$, which is both a cartesian product and coproduct, such that the following diagram commutes



In other words,

$$p_A i_A = 1, \quad p_A i_B = 0,$$
$$p_B i_A = 0, \quad p_B i_B = 1,$$

where p_A, p_B, i_A, i_B are the standard projections and injections belonging to the product/coproduct structures, the morphism 0 is the unique morphism that factors through the zero object, and 1 is the identity morphism.

If a category $\mathcal{A}$ has a zero object and biproducts of all pairs of objects, then we say that $\mathcal{A}$ is *semi-additive*. Some standard examples of semi-additive categories are: commutative monoids, abelian groups, modules, and vector bundles over a fixed space.

Exercise 1. Let $\mathcal{T}$ be an algebraic theory.

(a) (5 pt) Show that if $\operatorname{Alg}\mathcal{T}$ is semi-additive, then $\mathcal{T}$ is semi-additive.

(b) (6 pt) Show that if $\mathcal{T}$ is semi-additive, then $\operatorname{Alg}\mathcal{T}$ is semi-additive.

Solution 1.

(a) The Yoneda embedding $Y_{\mathcal{T}} : \mathcal{T}^{\mathrm{op}} \to \operatorname{Alg}\mathcal{T}$ is an equivalence onto its image $\mathrm{im}(Y_{\mathcal{T}})$ (+1pt). The image is closed under coproducts by 1.13 (+1pt), and hence also under biproducts and hence products (+1pt)[1]. Hence $\mathrm{im}(Y_{\mathcal{T}})$ is semi-additive and hence $\mathcal{T}^{\mathrm{op}}$ is, and hence $\mathcal{T}$ is (+2pt).

(b) The Yoneda embedding $Y_{\mathcal{T}}$ preserves both the initial and terminal objects (by 1.13), hence $\operatorname{Alg}\mathcal{T}$ has a zero object (+1pt). Let $A, B \in \mathrm{ob}(\operatorname{Alg}\mathcal{T})$ and define their biproduct $A \oplus B$ by

$$A \oplus B = \operatorname*{colim}_{\substack{(a,\alpha)\in \mathrm{El}A \\ (b,\beta)\in \mathrm{El}B}} Y_{\mathcal{T}}(a) \oplus Y_{\mathcal{T}}(b),$$

(+2pt). Now since $\mathrm{El}A \times \mathrm{El}B$ is sifted, $A \oplus B$ is both a product and a coproduct (+2pt). The equations $p_A i_A = 1, p_A i_B = 0, p_B i_A = 0, p_B i_B = 1$, follow by functoriality (+1pt).

[1] The original answer model was slightly incorrect about proving that the image is closed under products. We will not subtract points if the student does not prove that the image is closed under products.