Seminar Logic and Foundations of Computing
Homework 3

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Problem 1. Show that every colimit can be expressed as a filtered colimit of finite (finitely generated) colimits, viz. given a diagram $D : \mathcal{D} \to \mathcal{C}$ with $\mathcal{C}$ cocomplete and $\mathcal{D}$ small, $\operatorname{colim} D$ can be constructed as the filtered colimit of the diagram of all $\operatorname{colim} D'$, where $D' : \mathcal{D}' \to \mathcal{D}$ ranges over all domain restrictions of $\mathcal{D}$ to finitely generated subcategories $\mathcal{D}'$ of $\mathcal{D}$.

Solution. Let $\mathcal{C}$ be cocomplete, and let $D : \mathcal{D} \to \mathcal{C}$ be a small diagram. Then we have a diagram $F : \mathrm{FinSub}(\mathcal{D}) \to \mathcal{C}$ mapping finitely generated subcategories $\mathcal{D}'$ of $\mathcal{D}$ to their respective colimits, and mapping an inclusion morphism $f : \mathcal{D}' \to \mathcal{D}''$ to the factorisation of $\operatorname{colim} D''$ through $\operatorname{colim} D'$. Then we have to show a few things: We have to show that the category $\mathrm{FinSub}(\mathcal{D})$ is filtered, and we need to show that indeed these colimits are the same.

1. The category of finite subcategories is filtered. We can see this by using Lemma 2.19 ($3 \implies 1$). Firstly, note that the empty category is finitely generated, and therefore a subcategory of $\mathcal{D}$, Meaning that $\mathrm{FinSub}(\mathcal{D})$ is nonempty. Secondly, every pair of finitely generated subcategories $\mathcal{D}', \mathcal{D}''$ has a span given by the subcategory generated by the union of their objects and the union of their morphisms. Then clearly, $\mathcal{D}'$ and $\mathcal{D}''$ are subcategories, meaning we have inclusion morphisms which are a span. Thirdly, when we have two parallel morphisms $\mathcal{D}' \xrightarrow{f,g} \mathcal{D}''$, we have a morphism equalizing them: The identity on $\mathcal{D}''$, since inclusions are identical in this category.

2. To show that the colimits are the same, we will show that either colimit forms a cone over the others diagram. First, for each finitely generated subdiagram D' of D, clearly we have a cone of the colimit of D over D'. As a result, we find that there are maps $\operatorname{colim}_{d \in \mathcal{D}'} D' \to \operatorname{colim}_{d \in \mathcal{D}} D$ which factor all maps p_d (see diagram below) for $d \in \mathcal{D}'$. This creates a cone over F, since for any inclusion in $\mathrm{FinSub}(\mathcal{D})$, our choice of morphism must commute by virtue of forming a cone over the full diagram of D. As a result, we find that there is a unique map from $\operatorname{colim} F$ to $\operatorname{colim} D$.

$$
\begin{tikzcd}
 & \operatorname{colim} D \arrow[rrr, dashed, bend left=10] & & & \operatorname{colim} F \arrow[lll, dashed, bend left=10] & & \\
D(d) \arrow[ur, "p_d"] \arrow[rr, "D(f)"'] & & D(d') \arrow[ul, "p'_d"'] & \operatorname{colim} D' \arrow[ur, "\rho_{\operatorname{colim} D'}"] \arrow[rr] & & \operatorname{colim} D'' \arrow[ul, "\rho_{\operatorname{colim} D''}"'] & \\
 & & & D|_{\mathcal{D}'}(l) \arrow[u, "p_l^{\mathcal{D}'}"] \arrow[r] & D|_{\mathcal{D}'}(l') \arrow[ul, "p_{l'}^{\mathcal{D}'}"'] & D|_{\mathcal{D}''}(m) \arrow[u, "p_m^{\mathcal{D}''}"] \arrow[r] & D|_{\mathcal{D}''}(m') \arrow[ul, "p_{m'}^{\mathcal{D}''}"']
\end{tikzcd}
$$

Second, we will show that the colimit of F forms a cone over D: for every object $d \in \mathcal{D}$, form the finitely generated subcategory of $\mathcal{D}$ containing just that object. Since clearly the colimit of this diagram is equivalent to itself, we find that $\rho_d : Dd \to \operatorname{colim} F$ (up to isomorphism). This is indeed a cone, since for any morphism $f : d \to d'$, we have inclusion maps from the finitely generated subcategories $\{d\}$ and $\{d'\}$ to the finitely generated subcategory corresponding with the graph $\{d \xrightarrow{f} d'\}$. These inclusion maps give rise to morphisms from $\operatorname{colim}\{Dd\}$ and $\operatorname{colim}\{Dd'\}$ to $\operatorname{colim}\{Dd \xrightarrow{Df} Dd'\}$ which commute with the injections $\rho_{\{Dd\}}$, $\rho_{\{Dd'\}}$ and$\rho_{\{Dd \xrightarrow{Df} Dd'\}}$, proving that indeed the morphisms out of the singleton diagrams form a cone over D, giving us a map $\operatorname{colim} D \to \operatorname{colim} F$.

Finally, since the mappings of cones compose to precisely the identity on both sides, we find that their respective factorings must form an equivalence, completing our proof. $\square$