Seminar Logic and Foundations of Computing
Homework 2

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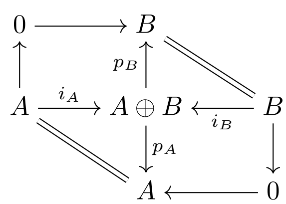
An object 0 that is both initial and terminal is called a *zero object*. In a category with a zero object, a *biproduct* of objects A and B is an object $A \oplus B$, which is both a cartesian product and coproduct, such that the following diagram commutes



In other words,

$$p_A i_A = 1, \quad p_A i_B = 0,$$
$$p_B i_A = 0, \quad p_B i_B = 1,$$

where p_A, p_B, i_A, i_B are the standard projections and injections belonging to the product/coproduct structures, the morphism 0 is the unique morphism that factors through the zero object, and 1 is the identity morphism.

If a category $\mathcal{A}$ has a zero object and biproducts of all pairs of objects, then we say that $\mathcal{A}$ is *semi-additive*. Some standard examples of semi-additive categories are: commutative monoids, abelian groups, modules, and vector bundles over a fixed space.

Exercise 1. Let $\mathcal{T}$ be an algebraic theory.

(a) (5 pt) Show that if $\operatorname{Alg}\mathcal{T}$ is semi-additive, then $\mathcal{T}$ is semi-additive.

(b) (6 pt) Show that if $\mathcal{T}$ is semi-additive, then $\operatorname{Alg}\mathcal{T}$ is semi-additive.