Seminar Logic and Foundations of Computing
Homework 2

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February 20, 2025

Exercise 1. In this exercise we aim to prove the following Lemma, which is used in part (b) of the proof of 12.15 on page 118.

Lemma 1 *Let $D : I \to Set$ be a filtered diagram in Set. Let $(c_i : Di \to C)_{i \in I}$ be a colimiting cocone and let $Z \subseteq Di$ be some finite subset for some i. Then there is some $(f : i \to j) \in I$, some finite subset $Z' \subseteq Dj$ and some surjection $e : Z \to Z'$ such that the diagram*

$$\begin{tikzcd}
Z \arrow[rr, "e"] \arrow[d, hook] & & Z' \arrow[d, hook] \\
Di \arrow[rr, "Df"] \arrow[dr, "c_i"'] & & Dj \arrow[dl, "c_j"] \\
& C &
\end{tikzcd}$$

commutes and such that $c_j|_{Z'}$ is injective.

(a) (3 points) Prove that there is some $f : i \to j$ such that for all $z_1, z_2 \in Z$, we have $c_i(z_1) = c_i(z_2)$ implies $Df(z_1) = Df(z_2)$.

(b) (3 points) Now choose $Z' := Df(Z)$ and prove the lemma.

Exercise 2. In this exercise we will see two examples of strictly many-sorted algebraic categories, i.e. algebraic categories which aren't single-sorted.

(a) (2 points) Prove that the category Set $\times$ Set (where morphisms are pairs of functions) is not single-sorted.

(b) (3 points) A category is called *trapped* if it contains two objects A, B such that there is no morphism $A \to B$. Let $\mathcal{C}$ be a small trapped category. Show that there exist a non-trivial splitting C_1, C_2 of objects of $\mathcal{C}$, such that for each pair of objects $c \in C_1, c' \in C_2$, there is no morphism $c \to c'$.

(c) (5 points) Let $\mathcal{C}$ again be a small trapped category, and let $\mathcal{A}$ be an algebraic category where the initial and terminal objects are not the same. Prove that the category $[\mathcal{C}, \mathcal{A}]$ is a strictly-multisorted algebraic category.