TYPE THEORY HW 3

All problems are in dependent type theory. Use function extensionality and univalence where appropriate.

Exercise 1. Show that homotopy between two functions is an equivalence relation.

Exercise 2. Show that

$$\prod_{A,B:\mathcal{U}} (A \times B) \simeq (B \times A)$$

Exercise 3. Show that

$$\prod_{A,B,C:\mathcal{U}} (A \to B \to C) \simeq ((A \times B) \to C)$$

Exercise 4. Consider a dependent type $x : B \vdash E(x) : \mathcal{U}$, any $a, b : B$, and any $p : \mathtt{Id}_B(a, b)$, show that $E(a) \simeq E(b)$.

Exercise 5. Show that for any type T, there is an equivalence

$$(T \to \bot) \simeq (T \simeq \bot).$$

Exercise 6. Consider a type S.

(1) Show that

$$\sum_{x:S} \top \simeq S$$

(2) Show that

$$\sum_{x:S} \bot \simeq \bot$$

Exercise 7. Consider any dependent type $x : \mathbb{B} \vdash E(x) : \mathcal{U}$.

(1) Show that

$$\mathtt{Id}_{\mathbb{B}}(0, 1) \to \bot$$

This is difficult. You may want to assume it's true and solve the rest of the problem set first.

(2) Show for any $e_0 \in E(0)$, $e_1 : E(1)$ that

$$\mathtt{Id}_{\sum_{x:\mathbb{B}} E(x)}((0, e_0), (1, e_1)) \to \bot$$

(3) Show that

$$\mathtt{Id}_{\mathbb{B}}(0, 0) \simeq \top$$

This is also difficult. You may want to assume it's true and solve the rest of the problem set first.

(4) Show that for any $e, e' \in E(0)$

$$\mathtt{Id}_{\sum_{x:\mathbb{B}} E(x)}((0, e), (0, e') \simeq \mathtt{Id}_{E(0)}(e, e')$$

Exercise 8. The rules for the natural numbers are as follows.

$$\frac{}{\vdash \mathbb{N} : \mathcal{U}}\ \mathbb{N}\text{-FORM} \qquad \frac{}{\vdash 0 : \mathbb{N}}\ \mathbb{N}\text{-INTRO}_1 \qquad \frac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash \mathsf{succ}(n) : \mathbb{N}}\ \mathbb{N}\text{-INTRO}_2$$

$$\frac{\Gamma, x{:}\mathbb{N} \vdash C(x) : \mathcal{U} \qquad \Gamma \vdash c_0 : C(0) \qquad \Gamma, x{:}\mathbb{N}, y{:}C \vdash c_s(x, y) : C(\mathsf{succ}(x))}{\Gamma, n : \mathbb{N} \vdash i_{c_0, c_s}(n) : C(n)}\ \mathbb{N}\text{-ELIM}$$

$$\frac{\Gamma, x{:}\mathbb{N} \vdash C(x) : \mathcal{U} \qquad \Gamma \vdash c_0 : C(0) \qquad \Gamma, x{:}\mathbb{N}, y{:}C \vdash c_s(x, y) : C(\mathsf{succ}(x))}{\Gamma \vdash i_{c_0, c_s}(0) = c_0 : C(0)}\ \mathbb{N}\text{-COMP}_1$$

$$\frac{\Gamma, x{:}\mathbb{N} \vdash C(x) : \mathcal{U} \qquad \Gamma \vdash c_0 : C(0) \qquad \Gamma, x{:}\mathbb{N}, y{:}C \vdash c_s(x, y) : C(\mathsf{succ}(x))}{\Gamma, n : \mathbb{N} \vdash i_{c_0, c_s}(\mathsf{succ}(n)) = c_s(n, i_{c_0, c_s}(n)) : C(\mathsf{succ}(n))}\ \mathbb{N}\text{-COMP}_2$$

(1) Define addition on $\mathbb{N}$.

(2) Prove that addition is commutative.