Higher mathematics via type theory

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15 March 2022

Outline

1. Background on type theory and univalent foundations
2. The univalence principle
3. Directed homotopy type theory
4. Summary

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3 Directed homotopy type theory

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Type theory's beginnings

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  - As a self-sufficient foundation of mathematics
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First-order logic

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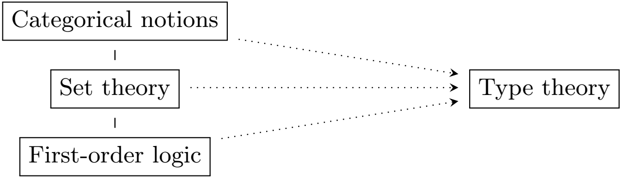

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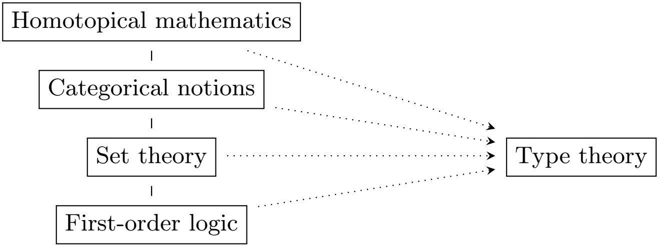

What does type theory look like?

- In mathematics, statements look like the following:
  - Consider a natural number n. The sum $n + n$ is even.
  - Consider a space X. The cone on X is contractible.
- In type theory, we write this as
  - $n : \mathbb{N} \vdash e(n) : \mathsf{isEven}(n + n)$
  - $X : \mathsf{Spaces} \vdash c(X) : \mathsf{isContr}(CX)$
- Type theory provides:
  - natural numbers type $\mathbb{N}$
  - product type $A \times B$
  - sum type $A + B$
  - function type $A \to B$
  - a universe type Type
  - a type (!) of equalities $a =_A b$
  - etc

Interpretations of type theory into classical mathematics

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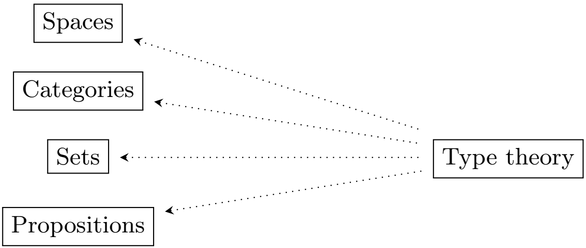

Interpretations of type theory into classical mathematics

Types	**Terms**	**Product**	**Equality**
Propositions	proofs	$\wedge$	$=$
Sets	elements	$\times$	$=$
Categories	objects	$\times$	$\cong$
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Univalence

- We've seen that equality in type theory can be interpreted as notions weaker than classical equality (e.g. isomorphism, paths).
- Voevodsky imported weakness for equality from the interpretation in spaces into type theory by imposing the *Univalence Axiom* (UA):

The canonical function $(A =_U B) \to (A \simeq B)$ is an equivalence of types, for any types A and B.

- UA is validated by the interpretation into spaces, but not into propositions, sets, or groupoids.
- Instead we **internalize** these notions.

Internalization of classical mathematics into type theory

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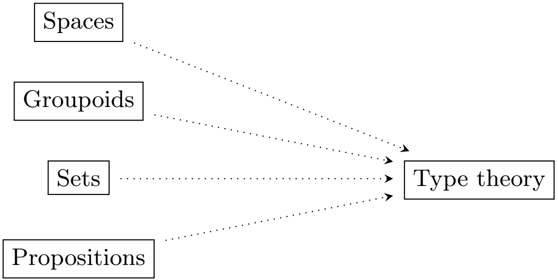

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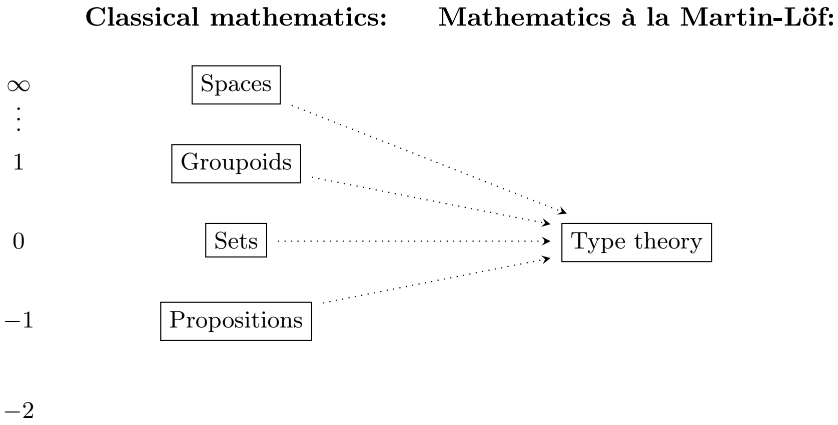

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- For types A, B which are structured sets (groups, rings, etc),

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so everything respects isomorphism of groups (or rings, etc).[2]

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- For *univalent* categories A, B,

$$(A =_{\mathrm{UCat}} B) \overset{UA}{\simeq} (A \simeq B) \simeq (A \simeq B)$$

so everything respects equivalence of univalent categories.[3]

[2] Coquand-Danielsson 2013

[3] Ahrens-Kapulkin-Shulman 2015

Univalent mathematics

- Voevodsky dreamt of ‘univalent mathematics’ in which

$$(A =_{\mathrm{D}} B) \simeq (A \simeq_{\mathrm{D}} B) \tag{$*$}$$

  where D is any type of mathematical object (propositions, sets, groups, categories, ∞-categories, etc) and $\simeq_{\mathrm{D}}$ is the appropriate notion of ‘sameness’ for that type of objects.
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- In our *Univalence Principle* project, we developed machinery which automatically produces ($*$) for a all D described by a signature (e.g. groups, categories, bicategories, opetopic categories, ...)
- In my *Directed homotopy type theory* project, I am developing a synthetic notion of directed equality that satisfies directed univalence.
$$(A \Rightarrow_{\mathrm{D}} B) \simeq (A \rightarrow_{\mathrm{D}} B)$$

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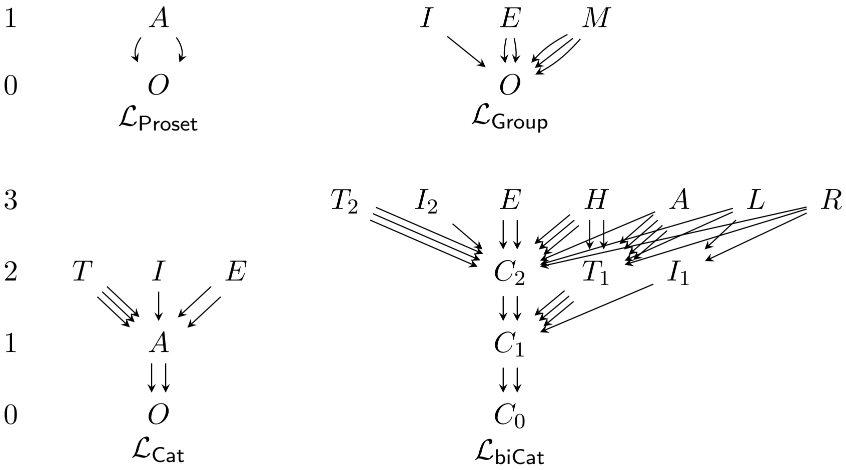
Signatures



1 A
0 O
$\mathcal{L}_{\mathsf{Proset}}$
I E M
O
$\mathcal{L}_{\mathsf{Group}}$
3
2 T I E
1 A
0 O
$\mathcal{L}_{\mathsf{Cat}}$
T_2 I_2 E H A L R
C_2 T_1 I_1
C_1
C_0
$\mathcal{L}_{\mathsf{biCat}}$

Structures

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- An $\mathcal{L}_{\mathsf{Cat}}$-structure $\mathcal{C}$ consists of:

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$\mathcal{L}_{\mathsf{Cat}}$

- $\mathcal{C}O : \mathsf{Type}$
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- Then we add axioms.

Level-wise equivalence

Proposition

For two $\mathcal{L}$-structures S, T,

$$(S =_{\mathcal{L}-\mathsf{Str}} T) \simeq (S \cong_{\mathcal{L}-\mathsf{Str}} T)$$

where $\cong_{\mathcal{L}-\mathsf{Str}}$ denotes levelwise equivalence.

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And is it appropriate to call $\mathcal{C}, \mathcal{D}$ categories?

Indiscernibility

Definition

Given an $\mathcal{L}$-structure M, and an object S of $\mathcal{L}$, we say that two elements $x, y : MS$ are *indiscernible* if substituting x for y in any object of $\mathcal{L}$ that depends on (i.e. object with a morphism to) S produces equivalent types.

Definition

An $\mathcal{L}$-structure M is *univalent* if for any object S of $\mathcal{L}$, and any $x, y : MS$, the type of indiscernibilities between x and y is equivalent to the type of equalities between x and y.

Univalent $\mathcal{L}_{\text{cat}}$ structures

Let $\mathcal{C}$ be a univalent $\mathcal{L}_{\text{cat}}$ structure.

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- Any two terms $x : \mathcal{C}O, f : \mathcal{C}A(x,x) \vdash i, j : \mathcal{C}I_x(f)$ are indiscernible.

$\rightarrow$ Each $\mathcal{C}I_x(f)$ is a proposition.

$\rightarrow$ Similarly, each $\mathcal{C}T_{x,y,z}(f,g,h)$, $\mathcal{C}E_{x,y}(f,g)$ is a proposition.

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→ Similarly, each $\mathcal{C}T_{x,y,z}(f,g,h)$, $\mathcal{C}E_{x,y}(f,g)$ is a proposition.

- In the axioms for a category, we have that E behaves like equality (is reflexive and a congruence for T, I, E.)

→ Univalence at A means that $f = g$ is equivalent to $\mathcal{C}E_{x,y}(f,g)$.

→ $\mathcal{C}A(x,y)$ is a set.

Univalent $\mathcal{L}_{\text{cat}}$ structures

- The indiscernibilities between $a, b : \mathcal{C}O$ consist of
  - $\phi_{x\bullet} : \mathcal{C}A(x,a) \cong \mathcal{C}A(x,b)$ for each $x : \mathcal{C}O$
  - $\phi_{\bullet z} : \mathcal{C}A(a,z) \cong \mathcal{C}A(b,z)$ for each $z : \mathcal{C}O$
  - $\phi_{\bullet\bullet} : \mathcal{C}A(a,a) \cong \mathcal{C}A(b,b)$
  - The following for all appropriate w, x, y, z, f, g, h:

$$\mathcal{C}T_{x,y,a}(f,g,h) \leftrightarrow \mathcal{C}T_{x,y,b}(f,\phi_{y\bullet}(g),\phi_{x\bullet}(h))$$
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  - $\phi_{\bullet\bullet} : \mathcal{C}A(a,a) \cong \mathcal{C}A(b,b)$
  - The following for all appropriate w, x, y, z, f, g, h:

$$\begin{aligned}
&\mathcal{C}T_{x,y,a}(f,g,h) \leftrightarrow \mathcal{C}T_{x,y,b}(f, \phi_{y\bullet}(g), \phi_{x\bullet}(h))\\
&\mathcal{C}T_{x,a,z}(f,g,h) \leftrightarrow \mathcal{C}T_{x,b,z}(\phi_{x\bullet}(f), \phi_{\bullet z}(g), h)\\
&\mathcal{C}T_{a,z,w}(f,g,h) \leftrightarrow \mathcal{C}T_{b,z,w}(\phi_{\bullet z}(f), g, \phi_{\bullet w}(h))\\
&\mathcal{C}T_{x,a,a}(f,g,h) \leftrightarrow \mathcal{C}T_{x,b,b}(\phi_{x\bullet}(f), \phi_{\bullet\bullet}(g), \phi_{x\bullet}(h))\\
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\end{aligned}$$

$$\begin{aligned}
&\mathcal{C}I_a(f) \leftrightarrow \mathcal{C}I_b(\phi_{\bullet\bullet}(f))\\
&\mathcal{C}E_{x,a}(f,g) \leftrightarrow \mathcal{C}E_{x,b}(\phi_{x\bullet}(f), \phi_{x\bullet}(g\\
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- But this an isomorphism in the usual categorical sense.

$\to$ Univalence at O means that $x = y$ is equivalent to $x \cong y$.

The right notion of equivalence

Main theorem

For two *univalent* $\mathcal{L}$-structures S, T,

$$(S =_{\mathcal{L}-\mathsf{Str}} T) \simeq (S \cong_{\mathcal{L}-\mathsf{Str}} T) \simeq (S \cong^*_{\mathcal{L}-\mathsf{Str}} T) \simeq (S \twoheadrightarrow T)$$

where $\cong^*_{\mathcal{L}-\mathsf{Str}}$ denotes levelwise equivalence up to indiscernbility and $\twoheadrightarrow$ denotes a very split surjective morphism.

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Very surjective morphisms of $\mathcal{L}_{\mathrm{cat}}$-structures

A *very surjective morphism* or *equivalence* $F : \mathcal{C} \simeq \mathcal{D}$ of $\mathcal{L}_{\mathrm{cat}}$-structures consists of surjections

- $FO : \mathcal{C}O \twoheadrightarrow \mathcal{D}O$
- $FA : \mathcal{C}A(x,y) \twoheadrightarrow \mathcal{D}A(Fx, Fy)$ for every $x, y : \mathcal{C}O$
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Summary

For every signature $\mathcal{L}$, we have

- a notion of structure,
- a notion of indiscernibility within each sort,
- a notion of univalent structures,
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The paper includes examples of

- †-categories,
- profunctors,
- bicategories,
- opetopic bicategories,
- ...

Current and future work

- Drop the splitness condition for certain structures.
- Extend to infinite structures.
- Formulate an analogue to the Rezk completion.
- Translate the theory into one about structures which can include explicit functions.
- Explore mathematics within examples.
- Give a model-theoretic account.

Outline

1. Background on type theory and univalent foundations
2. The univalence principle
3. Directed homotopy type theory
4. Summary

Directed type theory

- The **identity type** of type theory captures the paths of spaces, or the isomorphisms in a higher groupoid
- At the core of directed type theory is a **homomorphism type** which captures **directed** paths of **directed** spaces, or the morphisms of a higher category
- We are motivated by applications of directed topology to:
  - concurrent processes
  - robotics
  - rewriting
  - neuroscience

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higher groupoids

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higher categories

(undirected paths $\subseteq$ directed paths)

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higher categories
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directed spaces
(directed paths $\subseteq$ undirected paths)

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- We must also develop a directed version of homotopy theory, in particular a notion of **directed weak factorization system**

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Past work:

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Current work:

- Distinguishing different sorts of functoriality, or giving certain variables **modalities**, leads us to **modal type theory**
- We must also develop a directed version of homotopy theory, in particular a notion of **directed weak factorization system**

Future work:

- Develop notion of directed homotopy colimit, corresponding to higher inductive types of univalent foundations.
- Show that directed type theory is a conservative extension of type theory.
- Develop mathematics within directed type theory, oriented towards both theory and applications.

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- We envision a **Directed** univalence principle project one day.

Summary

- In the **Univalence principle** project, we develop higher structures within type theory.
- In the **Directed type theory** project, we modify type theory itself to encode higher categories and directed spaces.
- We envision a **Directed** univalence principle project one day.
- My research program focuses on
  - developing the most natural formalisms for homotopical and higher mathematics, and then
  - doing homotopical and higher mathematics therein.

Thank you!